§6.2 (Part 2): Some Important Formulas

1.] Recall the two variable LP (Problem Set 3, Number 1). Below the LP is in standard form along with its partially filled out optimal tableau.

Maximize: $z = 2x_1 + 2.5x_2$

Subject to:
$$x_1 + 2x_2 + s_1 = 350$$
$$2x_1 + x_2 + s_2 = 400$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Row	Basic	z	x_1	x_2	s_1	s_2	RHS
0	z	1	0	0	1	1/2	550
1	x_1	0	1	0	-1/3	2/3	150
2	x_2	0	0	1	2/3	-1/3	100

a.) Using the set of basic variables that yield the optimal solution, identify the vectors $\boldsymbol{x}_{BV}$, $\boldsymbol{x}_{NBV}$, $\boldsymbol{c}_{BV}$, $\boldsymbol{c}_{NBV}$ and the matrices B and N.

$$\vec{x}_{BV} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{x}_{NBV} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$$\vec{c}_{BV} = \begin{bmatrix} 2 \\ 2.5 \end{bmatrix} \quad \vec{c}_{NBV} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b.) Compute the matrix $B^{-1}N$ and use it to fill in the columns of the non-basic variables in the optimal tableau.

$$\left[\begin{array}{cc|cc} \overbrace{1 & 2}^{B} & \overbrace{1 & 0}^{I_2} \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ \underbrace{0 & 1}_{I_2} & \underbrace{2/3 & -1/3}_{B^{-1}} \end{array}\right]$$

$$B^{-1}N = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$$

c.) Use the formula $\overline{c_j} = \boldsymbol{c}_{BV}^{T} B^{-1} \boldsymbol{a}_j - c_j$ to compute the Row 0 coefficients of the non-basic variables.

For $\underline{s_1}$:
$$\overline{c_3} = \begin{bmatrix} 2 & 5/2 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 = \begin{bmatrix} 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

For $\underline{s_2}$:
$$\overline{c_4} = \begin{bmatrix} 2 & 5/2 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = \begin{bmatrix} 1 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1/2$$

d.) Compute $B^{-1}\boldsymbol{b}$ to find the RHS of each constraint for the optimal set of basic variables.

$$B^{-1}\vec{b} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 350 \\ 400 \end{bmatrix} = \begin{bmatrix} -350/3 + 800/3 \\ 700/3 - 400/3 \end{bmatrix} = \begin{bmatrix} 150 \\ 100 \end{bmatrix}$$

e.) Use the formula $\boldsymbol{c}_{BV}^T B^{-1}\boldsymbol{b}$ to find the optimal RHS in Row 0.

$$\vec{c}_{BV}^{\,T} B^{-1}\vec{b} = \begin{bmatrix} 2 & 2.5 \end{bmatrix}\begin{bmatrix} 150 \\ 100 \end{bmatrix} = 150(2) + 100(2.5) = 550$$

f.) Suppose we changed the coefficient for x_2 in the objective function to $2.5+\Delta c_2$. Find the optimality range by computing the coefficients in Row 0 of the non-basic variables.

For s_1:

$$\vec{c}_3 = \begin{bmatrix} 2 & 5/2 + \Delta c_2 \end{bmatrix}\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0$$

$$= \begin{bmatrix} 1 + \tfrac{2}{3}\Delta c_2 & \tfrac{1}{2} - \tfrac{1}{3}\Delta c_2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= 1 + \tfrac{2}{3}\Delta c_2$$

For s_2:

$$\vec{c}_4 = \begin{bmatrix} 2 & 5/2 + \Delta c_2 \end{bmatrix}\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0$$

$$= \begin{bmatrix} 1 + \tfrac{2}{3}\Delta c_2 & \tfrac{1}{2} - \tfrac{1}{3}\Delta c_2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \tfrac{1}{2} - \tfrac{1}{3}\Delta c_2$$

- In order for x_1, x_2 to remain optimal, we must have

$$1 + \tfrac{2}{3}\Delta c_2 \ge 0$$
$$\tfrac{1}{2} - \tfrac{1}{3}\Delta c_2 \ge 0$$

$$\Rightarrow \quad \Delta c_2 \ge -\tfrac{3}{2}, \quad \Delta c_2 \le \tfrac{3}{2}$$

$$\Rightarrow \quad \underbrace{\boxed{-\tfrac{3}{2} \le \Delta c_2 \le \tfrac{3}{2}}}_{\text{optimality range.}}$$