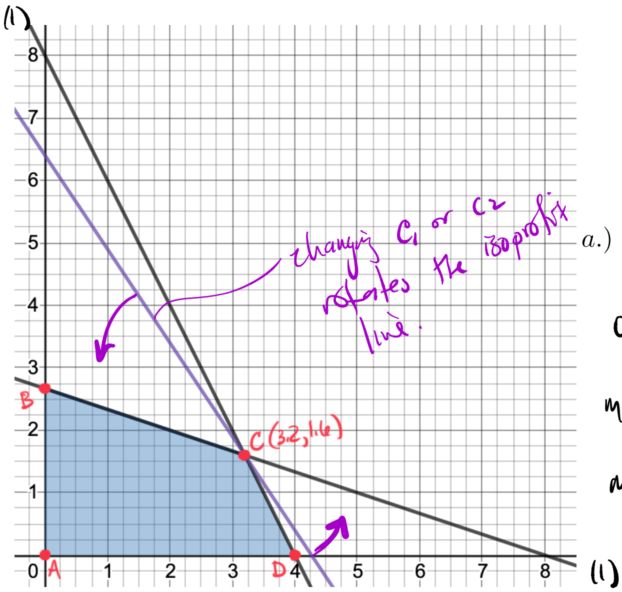
§6.1: Graphical Introduction to Sensitivity Analysis

1.] JOBCO manufactures two products on two machines. A unit of product 1 requires 2 hrs on machine 1 and 1 hor on machine 2. For product 2, one unit requires 1 hr on machine 1 and 3 hrs on machine 2. The revenues per unit of products 1 and 2 are \$30 and \$20 respectively. The total daily processing time available for each machine is 8 hrs. The LP and the feasible region with optimal solution are provided below.



Maximize: $z = 30x_1 + 20x_2$ obj. fun

Subject to: $2x_1 + x_2 \leq 8$ (machine 1) (1)

$x_1 + 3x_2 \leq 8$ (machine 2) (2)

$x_1, x_2 \geq 0$

a.) Write down the slope of the objective function and the two constraints.

obj fun: $x_2 = -\frac{30}{20}x_1 + \frac{z}{20} \Rightarrow$ slope $= -\frac{3}{2}$

machine 1: $x_2 = -2x_1 + 8 \Rightarrow$ slope $= -2$

machine 2: $x_2 = -\frac{1}{3}x_1 + \frac{8}{3} \Rightarrow$ slope $= -\frac{1}{3}$

b.) Suppose the revenue per unit of product 1 is c_1 making the objective function $z = c_1x_1 + 20x_2$. For what values of c_1 will the point $(3.2, 1.6)$ still be optimal?

In order for C to remain optimal, the isoprofit line must lie "in between" the two constraints. Hence,

1.) $-\frac{c_1}{20} < -\frac{1}{3} \Rightarrow c_1 > \frac{20}{3}$

2.) $-\frac{c_1}{20} > -2 \Rightarrow c_1 < 40$

$\boxed{\frac{20}{3} < c_1 < 40}$ optimality Range for c_1

c.) Suppose $z = c_1x_1 + c_2x_2$. Find the optimality range for the ratio ~~c_2/c_1~~ c_1/c_2. If $c_1 = 35$ and $c_2 = 25$, is $(3.2, 1.6)$ still optimal? What is the new value of z?

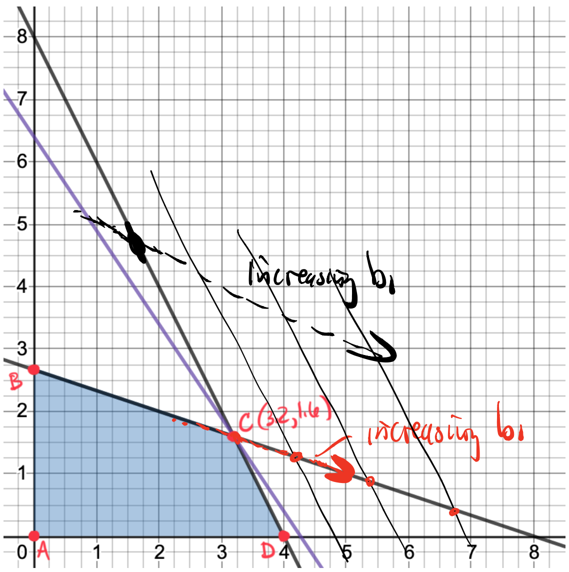
1.) $-\frac{c_1}{c_2} < -\frac{1}{3} \Rightarrow \frac{c_1}{c_2} > \frac{1}{3}$

2.) $-\frac{c_1}{c_2} > -2 \Rightarrow \frac{c_1}{c_2} < 2$

$\frac{1}{3} < \frac{c_1}{c_2} < 2$

$\frac{c_1}{c_2} = \frac{35}{25} = \frac{7}{5} \Rightarrow \frac{1}{3} < \frac{7}{5} < 2$ ✓

Max $z = 35(3.2) + 25(1.6) = 112 + 40 = 152$



c.) Suppose changes are made in machine 1 hour capacity. Redefine the first constraint to be $2x_1+x_2 \leq b_1$. Determine the feasibility range of b_1 that will still yield the same set of optimal basic variables.

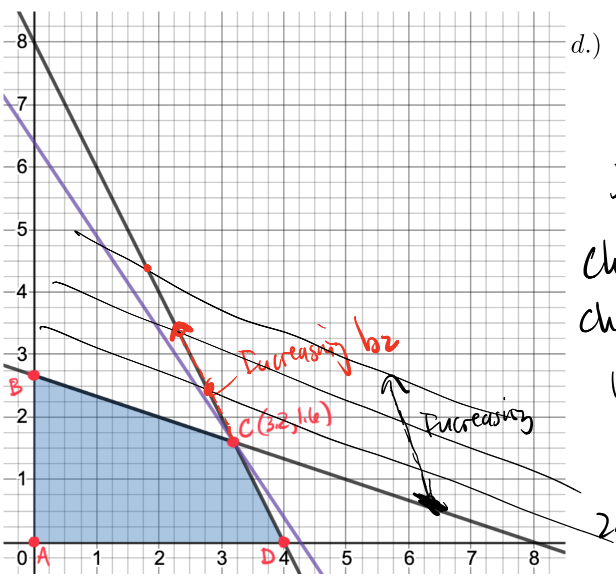
$x_2 = -2x_1 + b_1$ ← varying b_1 changes y-coordinate

Changing b_1 not only changes obj. function value, it changes the optimal values of x_1 and x_2.

1.) Upper bound: $x_1 = 8$, $x_2 = 0$ $\Rightarrow$ $0 = -2(8) + b_1$ $\Rightarrow$ $b_1 = 16$ ← x_2 would leave basis

2.) Lower bound: $x_1 = 0$, $x_2 = \frac{8}{3}$ $\Rightarrow$ $\frac{8}{3} = -2(0) + b_1$ $\Rightarrow$ $b_1 = \frac{8}{3}$ ← x_1 would leave basis

Feasibility Range for b_1: $\boxed{\frac{8}{3} < b_1 < 16}$



d.) Suppose changes are made in machine 2 hour capacity. Redefine the second constraint to be $x_1 + 3x_2 \leq b_2$. Determine the feasibility range of b_2 that will still yield the same set of optimal basic variables.

$x_2 = -\frac{1}{3}x_1 + \frac{b_2}{3}$ ← varying b_2 changes y-coordinate

Changing b_2 not only changes obj. function value, it changes the optimal values of x_1 and x_2.

1.) Upper bound: $x_1 = 0$, $x_2 = 8$ $\Rightarrow$ $8 = -\frac{1}{3}(0) + \frac{b_2}{3}$ $\Rightarrow$ $b_2 = 24$ ← x_1 would leave basis

2.) Lower bound: $x_1 = 4$, $x_2 = 0$ $\Rightarrow$ $0 = -\frac{1}{3}(4) + \frac{b_2}{3}$ $\Rightarrow$ $b_2 = 4$ ← x_2 would leave basis

Feasibility Range for b_2: $\boxed{4 < b_2 < 24}$

e.) What are the shadow prices for machine 1 and machine 2 capacities?

b_1:

$$2x_1 + x_2 \leq 8 + \Delta b_1$$
$$x_1 + 3x_2 \leq 8$$
$$\Rightarrow \quad x_1 = \frac{32 + 6\Delta b_1}{10}, \quad x_2 = \frac{8 - \Delta b_1}{5}$$

$$Z = 30x_1 + 20x_2$$

New Z:

$$\Rightarrow Z = 3(32 + 6\Delta b_1) + 4(8 - \Delta b_1)$$
$$\Rightarrow Z = 128 + \underbrace{14}_{\text{shadow price}}\Delta b_1$$

As long as $8 + \Delta b_1$ is within the feasibility range i.e.

$$\frac{8}{3} < 8 + \Delta b_1 < 16 \Rightarrow -\frac{16}{3} < \Delta b_1 < 8$$

the optimal solution is

$$x_1 = 3.2 + .6\Delta b_1$$
$$x_2 = 1.6 - .2\Delta b_1$$

and the obj. fun. value changes by 14 units per Δb_1 units change in b_1.