§6.2 (Part 1): Some Important Formulas

1.] Reddy Mikks Company: Recall the Reddy Mikks company once again (Worksheet 4.5 Part 1). Below the LP is in standard form along with its optimal tableau.

Maximize: $z = 5x_1 + 4x_2$

Subject to:
$$\begin{aligned} 6x_1 + 4x_2 + s_1 &= 24 \\ x_1 + 2x_2 + s_2 &= 6 \\ -x_1 + x_2 + s_3 &= 1 \\ x_2 + s_4 &= 2 \\ x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0 \end{aligned}$$

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
0	z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
1	x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
2	x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
3	s_3	0	0	0	$\frac{9}{24}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
4	s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Write down the matrix A and the vectors $\boldsymbol{x}$, $\boldsymbol{c}$, and $\boldsymbol{b}$. Construct the vectors $\boldsymbol{x}_{BV}$, $\boldsymbol{x}_{NBV}$, $\boldsymbol{c}_{BV}$, $\boldsymbol{c}_{NBV}$ and the matrices B and N using the set of basic variables that yield the optimal solution. Write the system in the alternative form.

- $A = \begin{bmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 24 \\ 6 \\ 1 \\ 2 \end{bmatrix}$

- $BV = \{1, 2, 5, 6\} \Rightarrow \vec{x}_{BV} = \begin{bmatrix} x_1 \\ x_2 \\ s_3 \\ s_4 \end{bmatrix}$, $\vec{c}_{BV} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \end{bmatrix}$

  $NBV = \{3, 4\} \Rightarrow \vec{x}_{NBV} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, $\vec{c}_{NBV} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- $B = \begin{bmatrix} 6 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ $\quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

- Maximize $z = \overbrace{\begin{bmatrix} 5 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_3 \\ s_4 \end{bmatrix}}^{\vec{c}_{BV}^{\,T} \vec{x}_{BV}} + \overbrace{\begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}}^{\vec{c}_{NBV}^{\,T} \vec{x}_{NBV}}$

  Subject to $\underbrace{\begin{bmatrix} 6 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_3 \\ s_4 \end{bmatrix}}_{B \vec{x}_{BV}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}}_{N \vec{x}_{NBV}} = \underbrace{\begin{bmatrix} 24 \\ 6 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b}}$ $\quad \vec{x}_{BV}, \vec{x}_{NBV} \geq \vec{0}$

2.] Find B^{-1} and confirm that $\boldsymbol{x}_{BV} + B^{-1}N\boldsymbol{x}_{NBV} = B^{-1}\boldsymbol{b}$ represents the optimal tableau.

$$\left[\begin{array}{cccc|cccc} \overbrace{6 & 4 & 0 & 0}^{B} & \overbrace{1 & 0 & 0 & 0}^{I_4} \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\frac{1}{6}R_1$, $-\frac{1}{6}R_1+R_2$, $\frac{1}{6}R_1+R_3$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 2/3 & 0 & 0 & 1/6 & 0 & 0 & 0 \\ 0 & 4/3 & 0 & 0 & -1/6 & 1 & 0 & 0 \\ 0 & 5/3 & 1 & 0 & 1/6 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\frac{3}{4}R_2$, $-\frac{3}{4}R_2+R_4$, $-\frac{5}{3}R_2+R_3$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 2/3 & 0 & 0 & 1/6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/8 & 3/4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3/8 & -5/4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1/8 & -3/4 & 0 & 1 \end{array}\right]$$

$-\frac{2}{3}R_2+R_1$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/4 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/8 & 3/4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3/8 & -5/4 & 1 & 0 \\ \underbrace{0 & 0 & 0 & 1}_{I_4} & \underbrace{1/8 & -3/4 & 0 & 1}_{B^{-1}} \end{array}\right]$$

$$B^{-1}N = \begin{bmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & -1/2 \\ -1/8 & 3/4 \\ 3/8 & -5/4 \\ 1/8 & -3/4 \end{bmatrix}$$

$$B^{-1}\vec{b} = \begin{bmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 24 \\ 6 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6-3 \\ -3+9/2 \\ 9-15/2+1 \\ 3-9/2+2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3/2 \\ 5/2 \\ 1/2 \end{bmatrix}$$

$$\vec{X}_{BV} + B^{-1}N\vec{X}_{NBV} = B^{-1}\vec{b}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ s_3 \\ s_4 \end{bmatrix} + \begin{bmatrix} 1/4 & -1/2 \\ -1/8 & 3/4 \\ 3/8 & -5/4 \\ 1/8 & -3/4 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3/2 \\ 5/2 \\ 1/2 \end{bmatrix}$$

which is equivalent to the optimal tableau.

3.] Confirm that Row 0 of the optimal tableau can be written as $z + \left(\boldsymbol{c}_{BV}^T B^{-1}N - \boldsymbol{c}_{NBV}^T\right)\boldsymbol{x}_{NBV} = \boldsymbol{c}_{BV}^T B^{-1}\boldsymbol{b}$.

$$\vec{C}_{BV}^T B^{-1}N - \vec{C}_{NBV}^T = \begin{bmatrix} 5 & 4 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/4 & -1/2 \\ -1/8 & 3/4 \\ 3/8 & -5/4 \\ 1/8 & -3/4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 \end{bmatrix}$$

$$\vec{C}_{BV}^T B^{-1}\vec{b} = \begin{bmatrix} 5 & 4 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 3/2 \\ 5/2 \\ 1/2 \end{bmatrix} = 15 + 6 = 21$$

Hence, $z + \begin{bmatrix} 3/4 & 1/2 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = 21 \Rightarrow z + \frac{3}{4}s_1 + \frac{1}{2}s_2 = 21$

which is equivalent to Row 0 in the optimal tableau.