§6.3 (Part 1): Sensitivity Analysis

1.] Giapetto's Workshop: Suppose x_1 and x_2 are the number of soldiers and trains, respectively, that Giapetto's produces and sells from his workshop. He sells each toy for a profit but is limited by two types of skilled labor hours: finishing (constraint 1) and carpentry (constraint 2). The third constraint is a demand constraint. The LP is below along with the optimal tableau.

Maximize Profit: $z = 3x_1 + 2x_2$

Subject to:
$$2x_1 + x_2 \le 100$$
$$x_1 + x_2 \le 80$$
$$x_1 \le 40$$
$$x_1, x_2 \ge 0$$

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
0	z	1	0	0	1	1	0	180
1	x_1	0	1	0	1	-1	0	20
2	x_2	0	0	1	-1	2	0	60
3	s_3	0	0	0	-1	1	1	20

a.) Show that as long as soldiers (x_1) contribute between \$2 and \$4 to profit, the current basis remains optimal. If soldier's contribute \$3.50 to profit, find the new optimal solution to the Giapetto problem.

- From the optimal tableau, we have $B^{-1}N = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ -1 & 1 \end{bmatrix}$. Since $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, it follows that $B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$. Hence, defining $c_1 = 3 + \Delta c_1$, we have

$$C_{BV}^T B^{-1} = \begin{bmatrix} 3+\Delta c_1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{array}{ccc} [\,1+\Delta c_1 & 1-\Delta c_1 & 0\,] \\ \uparrow & \uparrow & \\ s_1 & s_2 & \end{array}$$

- new Row 0 value for s_1: $\bar{c}_3 = 1 + \Delta c_1 \Rightarrow 1 + \Delta c_1 \ge 0 \Rightarrow \Delta c_1 \ge -1$
- new Row 0 value for s_2: $\bar{c}_4 = 1 - \Delta c_1 \Rightarrow 1 - \Delta c_1 \ge 0 \Rightarrow \Delta c_1 \le 1$

$\to -1 \le \Delta c_1 \le 1 \Rightarrow \boxed{2 \le c_1 \le 4}$

Optimality Range of c_1

- New optimal value $z_{opt} = \begin{bmatrix} 3.50 & 2 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ 60 \\ 20 \end{bmatrix} = 70 + 120 = \boxed{\$190}$

b.) Show that as long as trains (x_2) contribute between \$1.50 and \$3.00 to profit, then the current basis remains optimal.

- Let $c_2 = 2 + \Delta c_2$, then

$$C_{BV}^T B^{-1} = \begin{bmatrix} 3 & 2+\Delta c_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{array}{ccc} s_1 & s_2 & \\ [\,1-\Delta c_2 & 1+2\Delta c_2 & 0\,] \end{array}$$

- new Row 0 value for s_1: $\bar{c}_3 = 1 - \Delta c_2 \Rightarrow 1 - \Delta c_2 \ge 0 \Rightarrow \Delta c_2 \le 1$
- new Row 0 value for s_2: $\bar{c}_4 = 1 + 2\Delta c_2 \Rightarrow 1 + 2\Delta c_2 \ge 0 \Rightarrow \Delta c_2 \ge -\frac{1}{2}$

$\to -\frac{1}{2} \le \Delta c_2 \le 1 \Rightarrow \boxed{\frac{3}{2} \le c_2 \le 3}$

Optimality Range of c_2

c.) Show that if between 80 and 120 finishing hours are available, the current basis remains optimal. Find the new optimal solution to the Giapetto problem if 90 finishing hours are available.

- We let $b_1 = 100 + \Delta b_1$, so that the original RHS vector is $\vec{b} = \begin{bmatrix} 100+\Delta b_1 \\ 80 \\ 40 \end{bmatrix}$. The current basis will remain feasible (and optimal) as long as the new RHS of the optimal tableau (i.e. $B^{-1}\vec{b}$) consists of non-negative elements:

$$B^{-1}\vec{b} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 100+\Delta b_1 \\ 80 \\ 40 \end{bmatrix} = \begin{bmatrix} 100+\Delta b_1 - 80 \\ -100-\Delta b_1+160 \\ -100-\Delta b_1+80+40 \end{bmatrix} = \begin{bmatrix} 20+\Delta b_1 \\ 60-\Delta b_1 \\ 20-\Delta b_1 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 20+\Delta b_1 \geq 0 \\ 60-\Delta b_1 \geq 0 \\ 20-\Delta b_1 \geq 0 \end{matrix} \Rightarrow \begin{matrix} \Delta b_1 \geq -20 \\ \cancel{\Delta b_1 \leq 60} \\ \Delta b_1 \leq 20 \end{matrix} \quad -20 \leq \Delta b_1 \leq 20$$

$$\Rightarrow \boxed{80 \leq b_1 \leq 120}$$ Feasibility Range of b_1

If $b_1 = 90$, then $\Delta b_1 = 10$:

$$\Rightarrow \text{new opt. sol: } \vec{X}_{BV} = B^{-1}\vec{b} = \begin{bmatrix} 30 \\ 50 \\ 10 \end{bmatrix} \quad \text{New opt. value: } Z_{opt} = \vec{C}_{BV}^{T} B^{-1}\vec{b} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} 30 \\ 50 \\ 10 \end{bmatrix} = \boxed{\$190}$$

d.) Show that as long as the demand for soldiers is at least 20, the current basis remains optimal.

Let $\vec{b} = \begin{bmatrix} 100 \\ 80 \\ 40+\Delta b_3 \end{bmatrix}$, then we have

$$B^{-1}\vec{b} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 100 \\ 80 \\ 40+\Delta b_3 \end{bmatrix} = \begin{bmatrix} 100-80 \\ -100+160 \\ -100+80+40+\Delta b_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \\ 20+\Delta b_3 \end{bmatrix}$$

$$\Rightarrow 20+\Delta b_3 \geq 0 \Rightarrow \Delta b_3 \geq -20 \Rightarrow \boxed{b_3 \geq 20}$$

Feasibility Range of b_3