§4.8: Unbounded LPs

1.] Consider the following LP:

$$\text{Maximize:} \qquad z = 36x_1 + 30x_2 - 3x_3 - 4x_4$$

$$\begin{aligned} \text{Subject to:} \quad x_1 + x_2 - x_3 \quad &\leq 5 \\ 6x_1 + 5x_2 \quad - x_4 &\leq 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

a.) Solve the problem using the Simplex Method.

Row	Basic	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS	
0	z	1	-36	-30	3	4	0	0	0	
1	s_1	0	1	1	-1	0	1	0	5	$5/1 = 5$
2	s_2	0	$\boxed{6}$	5	0	-1	0	1	10	$10/6 = 5/3 \leftarrow$

Row	Basic	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS	
0'	z	1	0	0	3	-2	0	6	60	
1'	s_1	0	0	1/6	-1	$\boxed{1/6}$	1	-1/6	10/3	$\frac{10/3}{1/6} = 20 \leftarrow$
2'	x_1	0	1	5/6	0	-1/6	0	1/6	5/3	$\frac{5/3}{-1/6} = \otimes$

not Optimal!

$x_1 = 5,\ x_2 = 0,\ x_3 = 0,\ x_4 = 20$

$s_1 = 0,\ s_2 = 0$

Current $z = 100$.

Row	Basic	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
0"	z	1	0	2	-9	0	12	4	100
1"	x_4	0	0	1	-6	1	6	-1	20
2"	x_1	0	1	1	-1	0	1	0	5

b.) Suppose $x_3 = 1$. How would the variables x_1 and x_4 change? How would the objective function value change? Is this solution still feasible?

From the last tableau, we can write, for $x_3 \neq 0$:

$$\begin{aligned} x_1 &= 5 + x_3 \\ x_4 &= 20 + 6x_3 \end{aligned} \;\Rightarrow^{\text{if } x_3 = 1} \;\Rightarrow\; \begin{aligned} x_1 &= 6 \\ x &= 26 \end{aligned}$$

- Objective function value would increase by 9 units if $x_3 = 1$.
- Yes this new solution is still feasible since increasing x_3 still satisfies the first constraint.

c.) Let $\boldsymbol{x}$ be the the solution from the last iteration of the Simplex Method. Define the feasible solution $\boldsymbol{y} = \boldsymbol{d} + \boldsymbol{x}$ for some direction of unboundedness $\boldsymbol{d}$. Show that $\boldsymbol{c}^T\boldsymbol{d} > 0$.

Let $x_3 \neq 0$, then let $\vec{y}$ be the new feasible solution for when x_3 is introduced. We can write $\vec{y}$ as

$$\vec{c} = \begin{bmatrix} 36 \\ 30 \\ -3 \\ -4 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 5 + x_3 \\ 0 \\ x_3 \\ 20 + 6x_3 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 0 \\ 0 \\ 20 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}\text{, solution in last tableau}} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 6 \\ 0 \\ 0 \end{bmatrix}}_{\vec{d}\text{, direction of unboundedness}} \underbrace{x_3}_{\text{arbitrary constant}} \Rightarrow \vec{c}^T\vec{d} = \begin{bmatrix} 36 & 30 & -3 & -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 6 \\ 0 \\ 0 \end{bmatrix} = 36 + (-3) + (-24) = 9 > 0.$$

2.] Consider the following LP:

$$\text{Maximize:} \quad z = 20x_1 + 10x_2 + x_3$$

$$\begin{aligned} \text{Subject to:} \quad 3x_1 - 3x_2 + 5x_3 &\le 50 \\ x_1 \qquad\quad + x_3 &\le 10 \\ x_1 - x_2 + 4x_3 &\le 20 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

(a) By inspecting the constraints, determine the direction (x_1, x_2, or x_3) in which the solution space is unbounded. Define $\boldsymbol{d}$ and show $\boldsymbol{c}^T\boldsymbol{d} > 0$.

- Increasing x_2 would increase the objective function value while remaining feasible since constraints 1 and 3 will remain satisfied.
- In 3-dim space, the vector $\vec{d} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ is a direction of increase: $\vec{c}^T\vec{d} = [20 \;\; 10 \;\; 1]\begin{bmatrix}0\\1\\0\end{bmatrix} = 10 > 0$

(b) Without further computations, what can you conclude regarding the optimum objective value?

The optimum objective function value is unbounded and can be made arbitrarily large by increasing x_2.

3.] Consider the following minimization LP:

$$\text{Minimize:} \quad z = -x_1 - 3x_2$$

$$\begin{aligned} \text{Subject to:} \quad x_1 - 2x_2 &\le 4 \\ -x_1 \;\; + x_2 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Show that this LP is unbounded.

Row	Basic	z	x_1	x_2	s_1	s_2	RHS	
0	z	1	1	3	0	0	0	
1	s_1	0	1	-2	1	0	4	$4/-2 = \infty$
2	s_2	0	-1	$\boxed{1}$	0	1	3	$3/1 = 3$ ←

Row	Basic	z	x_1	x_2	s_1	s_2	RHS
0'	z	1	4	0	0	-3	-9
1'	s_1	0	-1	0	1	2	10
2'	x_1	0	-1	1	0	1	3

Unbounded LP. → z can be made arbitrarily small by increasing x_1.