§4.12: The Big M Method

1.] Consider the following LP:

$$\begin{aligned} \text{Minimize:} \quad & z = 4x_1 + x_2 \\ \text{Subject to:} \quad & 3x_1 + x_2 = 3 \\ & 4x_1 + 3x_2 \geq 6 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

a.) Convert this LP to standard form with additional artificial variables.

Standard Form:

Minimize $z = 4x_1 + x_2$

Subject to: $3x_1 + x_2 = 3$

$4x_1 + 3x_2 - e_2 = 6$

$x_1 + 2x_2 + s_3 = 4$

Standard form with artificial vars:

Minimize $z = 4x_1 + x_2 + Ma_1 + Ma_2$

Subject to: $3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - e_2 + a_2 = 6$

$x_1 + 2x_2 + s_3 = 4$

b.) Construct the initial tableau.

Row	Basic	z	x_1	x_2	e_2	a_1	a_2	s_3	RHS
0	z	1	-4	-1	0	$-M$	$-M$	0	0
1	a_1	0	3	1	0	1	0	0	3
2	a_2	0	4	3	-1	0	1	0	6
3	s_3	0	1	2	0	0	0	1	4

c.) Choose an appropriate value for M and eliminate the artificial variables from Row 0 by redefining the objective function in terms of the original decision and slack/excess variables. Then, construct the initial tableau again.

Substitute $a_1 = 3 - 3x_1 - x_2$ and $a_2 = 6 - 4x_1 - 3x_2 + e_2$ into z:

$z = 4x_1 + x_2 + M(3 - 3x_1 - x_2) + M(6 - 4x_1 - 3x_2 + e_2)$

$\Rightarrow z = (4 - 3M - 4M)x_1 + (1 - M - 3M)x_2 + Me_2 + (3M + 6M)$

$M = 100 \Rightarrow z = -696x_1 - 399x_2 + 100e_2 + 900$

Row	Basic	z	x_1	x_2	e_2	a_1	a_2	s_3	RHS	
0	z	1	696	399	-100	0	0	0	900	
1	a_1	0	3	1	0	1	0	0	3	$3/3 = 1$ ←
2	a_2	0	4	3	-1	0	1	0	6	$6/4 > 3/2$
3	s_3	0	1	2	0	0	0	1	4	$4/1 > 4$

∴ continue with Simplex Algorithm

2.] Consider the following LP:

$$\begin{aligned} \text{Minimize:} \quad & z = 4x_1 + 4x_2 + x_3 \\ \text{Subject to:} \quad & x_1 + x_2 + x_3 \le 2 \\ & 2x_1 + x_2 \le 3 \\ & 2x_1 + x_2 + 3x_3 \ge 3 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

a.) Rewrite the problem using the Big M Method. Eliminate all artificial variables from the objective function.

Standard form with artificial variables:

Minimize $z = 4x_1 + 4x_2 + x_3 + Ma_3$

Subject to: $x_1 + x_2 + x_3 + s_1 = 2$

$2x_1 + x_2 + s_2 = 3$

$2x_1 + x_2 + 3x_3 - e_3 + a_3 = 3$

$a_3 = 3 - 2x_1 - x_2 - 3x_3 + e_3$

$\Rightarrow z = 4x_1 + 4x_2 + x_3 + M(3 - 2x_1 - x_2 - 3x_3 + e_3)$

$\Rightarrow z = (4-2M)x_1 + (4-M)x_2 + (1-3M)x_3 + Me_3 + 3M$

$M = 100 \Rightarrow z = -196x_1 - 96x_2 - 299x_3 + 100e_3 + 300$

b.) Choose an appropriate value for M and solve the LP.

Row	Basic	z	x_1	x_2	x_3	e_3	s_1	s_2	a_3	RHS	
0	z	1	196	96	299	-100	0	0	0	300	
1	s_1	0	1	1	1	0	1	0	0	2	$2/1 = 2$
2	s_2	0	2	1	0	0	0	1	0	3	$3/0 = \infty$
3	a_3	0	2	1	3	-1	0	0	1	3	$3/3 = 1$ ←

Row	Basic	z	x_1	x_2	x_3	e_3	s_1	s_2	a_3	RHS
0'	z	1	$-10/3$	$-11/3$	0	$-1/3$	0	0	-299	1
1'	s_1	0	$1/3$	$2/3$	0	$1/3$	1	0	$-1/3$	1
2'	s_2	0	2	1	0	0	0	1	0	3
3'	x_3	0	$2/3$	$1/3$	1	$-1/3$	0	0	$1/3$	1

optimal!

Solution: $x_1 = 0, x_2 = 0, x_3 = 1$, $s_1 = 1, s_2 = 3, e_3 = 0$ Min $z = 1$