§4.13 (Part 2): The Two-Phase Simplex Method

1.] Consider the following LP:

$$\text{Maximize:} \quad z = 3x_1 + 2x_2 + 3x_3$$

$$\begin{aligned} \text{Subject to:} \quad 2x_1 + x_2 + x_3 &= 2 \\ x_1 + 3x_2 + x_3 &= 6 \\ 3x_1 + 4x_2 + 2x_3 &= 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

a.) Set up Phase I and solve it.

Maximize $w = a_1 + a_2 + a_3$

Subject to
$$\begin{aligned} 2x_1 + x_2 + x_3 + a_1 &= 2 \quad \Rightarrow \quad a_1 = 2 - 2x_1 - x_2 - x_3 \\ x_1 + 3x_2 + x_3 + a_2 &= 6 \quad \Rightarrow \quad a_2 = 6 - x_1 - 3x_2 - x_3 \\ 3x_1 + 4x_2 + 2x_3 + a_3 &= 8 \quad \Rightarrow \quad a_3 = 8 - 3x_1 - 4x_2 - 2x_3 \end{aligned}$$

Eliminate a_i from w: $w = (2 - 2x_1 - x_2 - x_3) + (6 - x_1 - 3x_2 - x_3) + (8 - 3x_1 - 4x_2 - 2x_3)$

$\Rightarrow w = 16 - 6x_1 - 8x_2 - 4x_3$

Row	Basic	w	x_1	x_2	x_3	a_1	a_2	a_3	RHS
0	w	1	6	8	4	0	0	0	16
1	a_1	0	2	$\boxed{1}$	1	1	0	0	2
2	a_2	0	1	3	1	0	1	0	6
3	a_3	0	3	4	2	0	0	1	8

$2/1 = 2$ ←
$6/3 = 2$ Tie!
$8/4 = 2$
(You could use Bland's Rule if you want.)

Optimal!

Row	Basic	w	x_1	x_2	x_3	a_1	a_2	a_3	RHS
0'	w	1	-10	0	-4	-8	0	0	0
1'	x_2	0	2	1	1	1	0	0	2
2'	a_2	0	$\boxed{-5}$	0	-2	-3	1	0	0
3'	a_3	0	-5	0	-2	-4	0	1	0

$1/\frac{1}{2} = 2$
$5/\frac{5}{2} = 2$ Tie!
$5/\frac{5}{2} = 2$

Although this is optimal, try to eliminate another artificial variable, using an artificial var. row as the pivot row.

Row	Basic	w	x_1	x_2	x_3	a_1	a_2	a_3	RHS
0"	w	1	0	0	0	-2	-2	0	0
1"	x_2	0	0	1	1/5	-1/5	2/5	0	2
2"	x_1	0	1	0	2/5	3/5	-1/5	0	0
3"	a_3	0	0	0	0	-1	-1	1	0
						Delete	Delete		

Two artificial vars, a_1 & a_2, are nonbasic, so we delete those rows. Leave a_3 in the tableau.

Note: If any other nonbasic vars would have had a neg. coeff. in Row 0, we can delete that column as well.

2.] Write down the associated LP for Phase II. Then construct the initial tableau and solve it.

Maximize: $Z = 3x_1 + 2x_2 + 3x_3$

Subject to

$$x_2 + \frac{1}{5}x_3 = 2 \implies x_2 = 2 - \frac{1}{5}x_3$$

$$x_1 + \frac{2}{5}x_3 = 0 \implies x_1 = -\frac{2}{5}x_3$$

$$a_3 = 0$$

Eliminate x_1 & x_2 from z:

$$z = 3\left(-\frac{2}{5}x_3\right) + 2\left(2 - \frac{1}{5}x_3\right) + 3x_3$$

$$\implies z = 4 - \frac{4}{5}x_3 - \frac{2}{5}x_3 + 3x_3$$

$$\implies z = 4 + \frac{7}{5}x_3$$

Redundant. Can be deleted.

Row	Basic	z	x_1	x_2	x_3	~~a_3~~	RHS
0	z	1	0	0	$-7/5$	~~0~~	4
1	x_2	0	0	1	$1/5$	~~0~~	2
2	x_1	0	1	0	$2/5$	~~0~~	0
~~3~~	~~a_3~~	~~0~~	~~0~~	~~0~~	~~0~~	~~1~~	0

Row	Basic	z	x_1	x_2	x_3	RHS	
0	z	1	0	0	$-7/5$	4	
1	x_2	0	0	1	$1/5$	2	$2/\frac{1}{5} = 10$
2	x_1	0	1	0	$\boxed{2/5}$	0	$0/\frac{2}{5} = 0$ ←

Optimal!

Row	Basic	z	x_1	x_2	x_3	RHS
0'	z	1	$7/2$	0	0	4
1'	x_2	0	$-1/2$	1	0	2
2'	x_3	0	$5/2$	0	1	0

Solution: $x_1 = 0,\ x_2 = 2,\ x_3 = 0,\ \max z = 4$