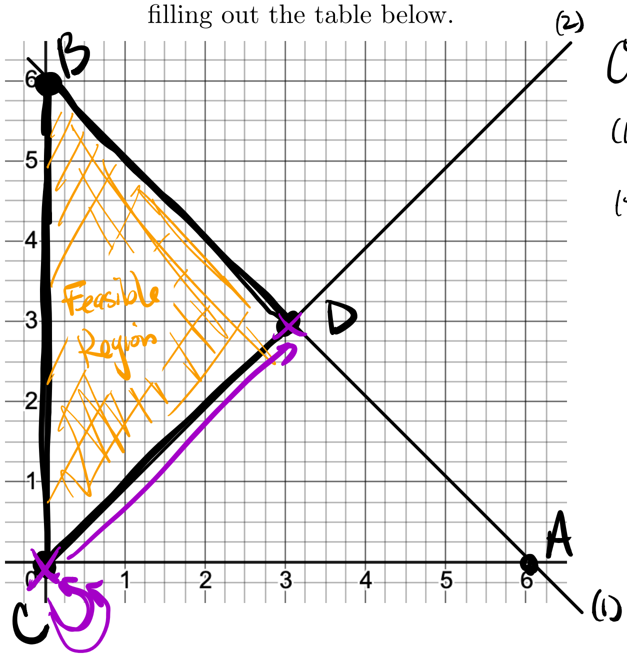
§4.11 (part 1): Degeneracy and Convergence of Simplex Method

1.] Consider the following two variable LP below:

$$\text{Maximize:} \quad z = 5x_1 + 2x_2$$

$$\text{Subject to:} \quad \begin{aligned} x_1 + x_2 &\le 6 \quad (1) \\ x_1 - x_2 &\le 0 \quad (2) \\ x_1, x_2 &\ge 0 \end{aligned}$$

Sketch the feasible space on the graph below, labeling the corner points of the feasible space. Convert the LP to standard form and identify all basic solutions with their corresponding objective values by filling out the table below.



Constraints:

(1) $y = 6 - x$

(2) $y = x$

Standard form:

$$x_1 + x_2 + s_1 = 6$$
$$x_1 - x_2 + s_2 = 0$$

$n=4$, $m=2$ > $_nC_m = {}_4C_2 = 6$ total basic sols

But there are only 4 intersections! (and 3 corner points)

Nonbasic Variables	Basic Variables	Basic Solution	Corner Point	Feasible?	Obj Value (z)	
x_1, x_2	s_1, s_2	$s_1 = 6, s_2 = 0$	C	Yes	0	(Degenerate)
x_1, s_1	x_2, s_2	$x_2 = 6, s_2 = 6$	B	Yes	12	
x_1, s_2	s_1, x_2	$x_2 = 0, s_1 = 6$	C	Yes	0	(Degenerate)
x_2, s_1	x_1, s_2	$x_1 = 6, s_2 = -6$	A	No	—	
x_2, s_2	x_1, s_1	$x_1 = 0, s_1 = 6$	C	Yes	0	(Degenerate)
s_1, s_2	x_1, x_2	$x_1 = 3, x_2 = 3$	D	Yes	21	optimal.

Cycling: It is possible that when choosing the next entering variable, the algorithm will select a bfs that is still at C instead of moving to another corner!

2.] Show that even though the initial tableau is not degenerate, later iterations may exhibit degeneracy.

$$\text{Maximize:} \quad z = 5x_1 + 3x_2$$

$$\text{Subject to:} \quad \begin{aligned} 4x_1 + 2x_2 &\le 12 \\ 4x_1 + x_2 &\le 10 \\ x_1 + x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
0	z	1	−5	−3	0	0	0	0
1	s_1	0	4	2	1	0	0	12
2	s_2	0	$\boxed{4}$	1	0	1	0	10
3	s_3	0	1	1	0	0	1	4

$12/4 = 3$
$10/4 = 5/2 \leftarrow$
$4/1 = 4$

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
0’	z	1	0	−7/4	0	5/4	0	25/2
1’	s_1	0	0	$\boxed{1}$	1	−1	0	2
2’	x_1	0	1	1/4	0	1/4	0	5/2
3’	s_3	0	0	3/4	0	−1/4	1	3/2

$2/1 = 2 \leftarrow$
$\frac{5/2}{1/4} = 10$ (Tie!)
$\frac{3/2}{1/4} = 2$

Degenerate!
$s_3 = 0$

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
0”	z	1	0	0	7/4	−1/2	0	16
1”	x_2	0	0	1	1	−1	0	2
2”	x_1	0	1	0	−1/4	1/2	0	2
3”	s_3	0	0	0	−3/4	$\boxed{1/2}$	1	0

$2/-1 = ⊗$
$2/\tfrac{1}{2} = 4$
$0/\tfrac{1}{2} = 0 \leftarrow$

Optimal!

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
0”’	z	1	0	0	1	0	1	16
1”’	x_2	0	0	1	−1/2	0	2	2
2”’	x_1	0	1	0	1/2	0	−1	2
3”’	s_2	0	0	0	−3/2	1	2	0

$$\boxed{\begin{array}{l} x_1 = 2,\ x_2 = 2,\ s_2 = 0 \\ s_1 = s_3 = 0 \\ \qquad \text{Max } z = 16 \end{array}}$$