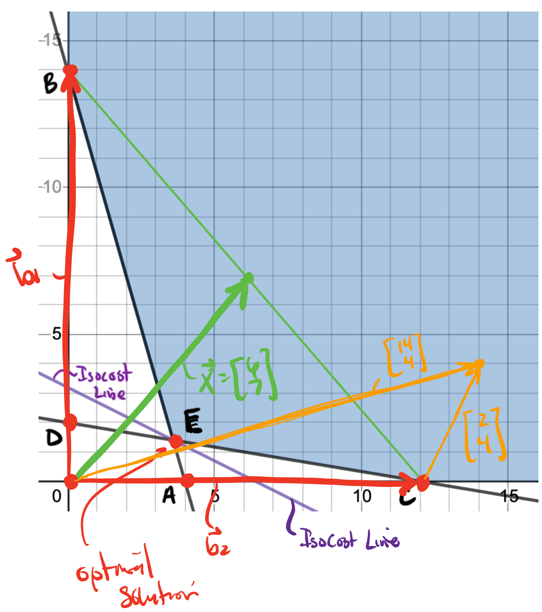
§4.3: Direction of Unboundedness

1.] Dorian Auto: In Example 2 of Chapter 3, the text formulates the LP for Dorian Auto manufacturers. The LP is given below in standard form:

$$\text{Minimize:} \quad z = 50x_1 + 100x_2$$

$$\begin{aligned} \text{Subject to:} \quad & 7x_1 + 2x_2 - e_1 = 28 \\ & 2x_1 + 12x_2 - e_2 = 24 \\ & x_1, x_2, e_1, e_2 \geq 0 \end{aligned}$$

Sketching the feasible region in two-dimensional space gives the following unbounded convex set:



a.) Show that the feasible solution corresponding to the point $(6, 7)$, call it $\boldsymbol{x}$, can be rewritten as a linear combination of the form $\boldsymbol{x} = \sigma_1 \boldsymbol{b}_1 + \sigma_2 \boldsymbol{b}_2$, where $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ correspond to the basic feasible solutions at the points $(0, 14)$ and $(12, 0)$, respectively, and $\sigma_1 + \sigma_2 = 1$.

Clearly: $\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 14 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 12 \\ 0 \end{bmatrix} \Rightarrow \sigma_1 = \frac{1}{2}, \sigma_2 = \frac{1}{2}.$

LP Vectors: If $x_1 = 6, x_2 = 7$, then $e_1 = 28$, $e_2 = 72$.
If $x_1 = 0, x_2 = 14$, then $e_1 = 0$, $e_2 = 144$.
If $x_1 = 12, x_2 = 0$, then $e_1 = 56$, $e_2 = 0$.

Then we have

$$\vec{x} = \begin{bmatrix} 6 \\ 7 \\ 28 \\ 72 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 14 \\ 0 \\ 144 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 12 \\ 0 \\ 56 \\ 0 \end{bmatrix}$$

2.] Show that the basic feasible solution with non-basic variables $e_1, e_2 = 0$ corresponds to corner point E by solving a particular two-dimensional linear system.

Let $e_1 = e_2 = 0$, then substituting into the standard form of the LP gives the following system

$$7x_1 + 2x_2 = 28$$
$$2x_1 + 12x_2 = 24$$

$$\Rightarrow \begin{array}{c} \frac{1}{7}\times \\ \frac{1}{2}\times \end{array} \left[\begin{array}{cc|c} 7 & 2 & 28 \\ 2 & 12 & 24 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} -1\times \\ \text{add} \end{array} \left[\begin{array}{cc|c} 1 & 2/7 & 4 \\ 1 & 6 & 12 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} \\ \frac{7}{40}\times \end{array} \left[\begin{array}{cc|c} 1 & 2/7 & 4 \\ 0 & 40/7 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2/7 & 4 \\ 0 & 1 & 7/5 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_2 = 7/5 \\ x_1 = -\frac{2}{7}x_2 + 4 \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = 7/5 \\ x_1 = -\frac{2}{5} + \frac{20}{5} \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = 7/5 \\ x_1 = 18/5 \end{cases}$$

Optimal Sol:
$x_2 = 7/5$, $x_1 = 18/5$

Min $z = 320$

3.] Consider the point $(14, 4)$ in the feasible space. This point is not a basic feasible solution, why? Determine the vector $\boldsymbol{x}$ that corresponds to this point. Represent this point as a combination of the form $\boldsymbol{x} = \boldsymbol{d} + \boldsymbol{b_2}$, where $\boldsymbol{b_2}$ is the vector corresponding to the basic feasible solution $(12, 0)$.

- $(14,4)$ is not a basic feasible solution because it is not a corner point. In other words, it is not an intersection of two constraints.
- From the graph, we can attain $\begin{bmatrix}14\\4\end{bmatrix}$ by adding $\begin{bmatrix}2\\4\end{bmatrix}$ to $\begin{bmatrix}12\\0\end{bmatrix}$, i.e. $\overbrace{\begin{bmatrix}14\\4\end{bmatrix}}^{\vec{x}} = \overbrace{\begin{bmatrix}2\\4\end{bmatrix}}^{\vec{d}} + \overbrace{\begin{bmatrix}12\\0\end{bmatrix}}^{\vec{b}_1}$.
- Let $x_1 = 14$, $x_2 = 4$, then $e_1 = 78$, $e_2 = 52$. Thus, $\vec{x} = \begin{bmatrix}14\\4\\28\\52\end{bmatrix}$.
- we already know that $\vec{b}_2 = \begin{bmatrix}12\\0\\56\\0\end{bmatrix}$. $\longrightarrow$ $\vec{x} = \begin{bmatrix}14\\4\\28\\52\end{bmatrix} = \begin{bmatrix}2\\4\\??\\??\end{bmatrix} + \begin{bmatrix}12\\0\\56\\0\end{bmatrix} \Rightarrow \vec{d} = \begin{bmatrix}2\\4\\22\\52\end{bmatrix}$

4.] Find the null space of the matrix A and show that the vector representing the direction of unboundedness, $\boldsymbol{d}$, from the previous question is indeed in the null space of the matrix.

Direction of unboundedness

- Null space of A is all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$. We have

$$\begin{array}{c} 2\times \\ 7\times \end{array}\left[\begin{array}{cccc|c} 7 & 2 & -1 & 0 & 0 \\ 2 & 12 & 0 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} 1\times \\ \end{array}\left[\begin{array}{cccc|c} 14 & 4 & -2 & 0 & 0 \\ 14 & 84 & 0 & -7 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} 20\times \\ \end{array}\left[\begin{array}{cccc|c} 14 & 4 & -2 & 0 & 0 \\ 0 & 80 & 2 & -7 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} \\ -1\times \end{array}\left[\begin{array}{cccc|c} 280 & 80 & -40 & 0 & 0 \\ 0 & 80 & 2 & -7 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} 1/280\times \\ 1/80\times \end{array}\left[\begin{array}{cccc|c} 280 & 0 & -42 & 7 & 0 \\ 0 & 80 & 2 & -7 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -3/20 & 1/40 & 0 \\ 0 & 1 & 1/40 & -7/80 & 0 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} d_1 = \frac{3}{20}d_3 - \frac{1}{40}d_4 \\ d_2 = -\frac{1}{40}d_3 + \frac{7}{80}d_4 \end{array}\right\}$$

d_3, d_4 are free

The null space of A is two-dimensional.

- Let $d_3 = 22$, $d_4 = 52$, then

$$d_1 = \frac{3}{20}(22) - \frac{1}{40}(52)$$
$$\Rightarrow d_1 = \frac{66 - 26}{20}$$
$$\Rightarrow d_1 = \frac{40}{20}$$
$$\Rightarrow d_1 = 2$$

$$d_2 = -\frac{1}{40}(22) + \frac{7}{80}(52)$$
$$\Rightarrow d_2 = \frac{-22 + 7(26)}{40}$$
$$\Rightarrow d_2 = \frac{160}{40}$$
$$\Rightarrow d_2 = 4$$

Thus, the direction of unboundedness is

$$\vec{d} = \begin{bmatrix}2\\4\\22\\52\end{bmatrix} \in Null(A)$$

which resides in the null space of A!