§4.7: Alternative Optimal Solutions

1.] Find two optimal solutions to the two-variable LP problem using the Simplex Method:

$$\text{Maximize:} \quad z = 4x_1 + 2x_2$$

$$\text{Subject to:} \quad \begin{aligned} 2x_1 + x_2 &\leq 100 \\ x_1 + x_2 &\leq 80 \\ x_1 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
0	z	1	-4	-2	0	0	0	0
1	s_1	0	2	1	1	0	0	100
2	s_2	0	1	1	0	1	0	80
3	s_3	0	$\boxed{1}$	0	0	0	1	40

$100/2 = 50$

$80/1 = 80$

$40/1 = 40 \leftarrow$

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
0'	z	1	0	-2	0	0	4	160
1'	s_1	0	0	$\boxed{1}$	1	0	-2	20
2'	s_2	0	0	1	0	1	-1	40
3'	x_1	0	1	0	0	0	1	40

$20/1 = 20 \leftarrow$

$40/1 = 40$

$40/0 = \otimes$

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
0"	z	1	0	0	2	0	0	200
1"	x_2	0	0	1	1	0	-2	20
2"	s_2	0	0	0	-1	1	$\boxed{1}$	20
3"	x_1	0	1	0	0	0	1	40

$20/-2 = \otimes$

$20/1 = 20 \leftarrow$

$40/1 = 40$

optimal!

Max $z = 200$

$x_1 = 40$, $x_2 = 20$

$s_1 = 0$, $s_2 = 20$, $s_3 = 0$

Row	Basic	z	x_1	x_2	s_1	s_2	s_3	RHS
0"'	z	1	0	0	2	0	0	200
1"'	x_2	0	0	1	-1	2	0	60
2"'	s_3	0	0	0	-1	1	1	20
3"'	x_1	0	1	0	1	-1	0	20

optimal!

Max $z = 200$

$x_1 = 20$, $x_2 = 60$

$s_1 = 0$, $s_2 = 20$, $s_3 = 0$

2.] In the previous example, denote $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ as the two optimal solutions. Show that $\boldsymbol{x} = \sigma_1\boldsymbol{b_1} + \sigma_2\boldsymbol{b}_2$, with $\sigma_1 + \sigma_2 = 1$, is also optimal.

$$\vec{b}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 40 \\ 20 \end{bmatrix}$$

$$\vec{b}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

$$\vec{x} = \sigma_1 \begin{bmatrix} 40 \\ 20 \end{bmatrix} + \sigma_2 \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 40\sigma_1 + 20\sigma_2 \\ 20\sigma_1 + 60\sigma_2 \end{bmatrix}$$

$$\begin{aligned} z &= 4x_1 + 2x_2 \\ &= 4(40\sigma_1 + 20\sigma_2) + 2(20\sigma_1 + 60\sigma_2) \\ &= 160\sigma_1 + 80\sigma_2 + 40\sigma_1 + 120\sigma_2 \\ &= 200\sigma_1 + 200\sigma_2 \\ &= 200(\sigma_1 + \sigma_2) \\ &= \boxed{200} \end{aligned}$$

Thus, $\vec{x}$ is also optimal with the same obj. fun. value.

3.] Consider an LP with objective function $z = \boldsymbol{c}^T\boldsymbol{x}$ and feasible region given by $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{x} \geq \boldsymbol{0}$. Suppose $\boldsymbol{b}_i$ for $i = 1, 2, \ldots, k$ are all optimal solutions. Show that any convex combination of these optimal solutions is also optimal.

Suppose z_{opt} is the optimal obj. fun. value for each $\vec{b}_i$, i.e. $z_{opt} = \vec{c}^T\vec{b}_i$ for all $i = 1, 2, \ldots, k$. Let $\vec{x} = \sum_{i=1}^{k} \sigma_i \vec{b}_i$ such that $\sum_{i=1}^{k} \sigma_i = 1$, then

$$\begin{aligned} \vec{c}^T\vec{x} &= \vec{c}^T\left(\sum_{i=1}^{k} \sigma_i \vec{b}_i\right) \\ &= \sum_{i=1}^{k} \sigma_i (\vec{c}^T \vec{b}_i) \\ &= \sum_{i=1}^{k} \sigma_i (z_{opt}) \\ &= z_{opt}\left(\sum_{i=1}^{k} \sigma_i\right) \\ &= z_{opt}(1) \\ &= z_{opt}. \end{aligned}$$

Hence, $\vec{x}$ is also optimal with the same obj. fun. value.

4.] Below are two partial tableaus for two separate maximization LPs. Are they optimal? What are the basic and nonbasic variables of each problem? Which LP has alternative optima?

Row	z	x_1	x_2	s_1	s_2	RHS
0	1	0	0	2	3	10
1	0	1	0	3	2	4
2	0	0	1	1	1	3

- Is optimal with unique solution because the coeffs in Row 0 for all NBVs are strictly positive.
- BVs: x_1, x_2. NBVs: s_1, s_2.
- Max $z = 10$, $x_1 = 4$, $x_2 = 3$

Row	z	x_1	x_2	s_1	s_2	RHS
0	1	0	0	0	2	2
1	0	1	0	-1	1	2
2	0	0	1	-2	3	3

- Is optimal, but alternative optima exist because there exists a 0 in Row 0 for the NBV s_1.
- BVs: x_1, x_2. NBVs: s_1, s_2
- Max $z = 2$, $x_1 = 2$, $x_2 = 3$