§3.10: Multi-Period Production/Inventory Models

1.] Monthly Production: A customer requires during the next four months, respectively, 50, 65, 100, and 70 units of a commodity (no backlogging is allowed). Production costs are \$5, \$8, \$4, and \$7 per unit during these months. The storage cost from one month to the next is \$2 per unit (assessed on ending inventory). It is estimated that each unit on hand at the end of month 4 could be sold for \$6. Formulate an LP that will minimize the net cost incurred in meeting the demands of the next four months.

Decision Variables: X_t = units made in month t $(t=1,2,3,4)$
i_t = inventory at the end of month t $(t=1,2,3,4)$

Objective Function: Minimize Cost, Costs = (Production) + (Storage) − (Sales)

$$\Rightarrow \quad Z = (5X_1 + 8X_2 + 4X_3 + 7X_4) + (2i_1 + 2i_2 + 2i_3 + 2i_4) - (6i_4)$$

Constraints: The only constraints come from the inventory variables:

(Inventory Requirements)

$$\begin{aligned} i_1 &= X_1 - 50 \\ i_2 &= i_1 + X_2 - 65 \\ i_3 &= i_2 + X_3 - 100 \\ i_4 &= i_3 + X_4 - 70 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X_1 \quad - i_1 \quad &= 50 \\ X_2 \quad + i_1 - i_2 &= 65 \\ X_3 \quad + i_2 - i_3 &= 100 \\ X_4 \quad + i_3 - i_4 &= 70 \end{aligned}$$

$$X_t,\ i_t \geq 0$$

Solution: From Excel:
$X_1 = 115$, $X_2 = 0$, $X_3 = 170$, $X_4 = 0$
$i_1 = 65$, $i_2 = 0$, $i_3 = 70$, $i_4 = 0$
Min Cost $Z = \$1525.00$

2.] Baking Cakes: James Beerd bakes cheesecakes and Black Forest cakes. During any month, he can bake at most 65 cakes. The costs per cake and the demands for cakes, which must be met on time, are listed in the table below. It costs \$0.50 to hold a cheesecake, and \$0.40 to hold a Black Forest cake, in inventory for a month. Formulate an LP to minimize the total cost of meeting the next three months' demands.

	Month 1		Month 2		Month 3	
Item	*Demand*	*Cost/Cake (\$)*	*Demand*	*Cost/Cake (\$)*	*Demand*	*Cost/Cake (\$)*
Cheesecake	40	3.00	30	3.40	20	3.80
Black Forest	20	2.50	30	2.80	10	3.40

Decision Variables:
X_{ct} = cheesecakes made in month t $(t=1,2,3)$
X_{bt} = black forest cakes made in month t $(t=1,2,3)$
i_{ct} = inventory of cheesecakes at end of month $(t=1,2,3)$
i_{bt} = inventory of black forest cakes at end of month $(t=1,2,3)$

Objective Function: Minimize Cost, Costs = (Cake Costs) + (Holding Costs)

$$Z = 3X_{c1} + 2.50X_{b1} + 3.40X_{c2} + 2.80X_{b2} + 3.80X_{c3} + 3.40X_{b3} + \cdots$$
$$.50(i_{c1} + i_{c2} + i_{c3}) + .40(i_{b1} + i_{b2} + i_{b3})$$

Constraints:

(Inventory Requirements)

$$\begin{aligned} i_{c1} &= X_{c1} - 40 \\ i_{c2} &= i_{c1} + X_{c2} - 30 \\ i_{c3} &= i_{c2} + X_{c3} - 20 \\ i_{b1} &= X_{b1} - 20 \\ i_{b2} &= i_{b1} + X_{b2} - 30 \\ i_{b3} &= i_{b2} + X_{b3} - 10 \end{aligned}$$

$$i_{ct}, i_{bt} \geq 0$$

(Max Quantity)

$$\begin{aligned} X_{c1} + X_{b1} &\leq 65 \\ X_{c2} + X_{b2} &\leq 65 \\ X_{c3} + X_{b3} &\leq 65 \end{aligned}$$

$$X_{ct}, X_{bt} \geq 0$$

Solution: From Excel:
$X_{c1} = 40$, $X_{c2} = 30$, $X_{c3} = 20$
$X_{b1} = 25$, $X_{b2} = 35$, $X_{b3} = 0$
$i_{c1} = i_{c2} = i_{c3} = 0$
$i_{b1} = 5$, $i_{b2} = 10$, $i_{b3} = 0$
Min Cost $Z = \$464.50$

3.] General Cars: During the next two months, General Cars must meet (on time) the following demands for trucks and cars: month 1 – 400 trucks, 800 cars; month 2 – 300 trucks, 300 cars. During each month, at most 1000 vehicles can be produced. Each truck uses 2 tons of steel, and each car uses 1 ton of steel. During month 1, steel costs $400 per ton; during month 2, steel costs $600 per ton. At most, 1500 tons of steel may be purchased each month (steel may only be used during the month in which it is purchased). At the beginning of month 1, 100 trucks and 200 cars are in inventory. At the end of each month, a holding cost of $150 per vehicle is assessed. A car gets 36 mpg, and each truck gets 22 mpg. During each month, the vehicles produced by the company must average at least 30 mpg. Formulate an LP to meet the demand and mileage requirements at minimum cost (include steel costs and holding costs).

Decision Variables:

X_{tj} = trucks produced in month j $(j=1,2)$

X_{cj} = cars produced in month j $(j=1,2)$

i_{tj} = inventory of trucks at end of month j $(j=0,1,2)$

i_{cj} = inventory of cars at end of month j $(j=0,1,2)$

S_{tj} = tons of steel used to make trucks in month j $(j=1,2)$

S_{cj} = tons of steel used to make cars in month j $(j=1,2)$

Note: $i_{t0} = 100$, $i_{c0} = 200$

Objective Function: Minimize cost while meeting demand and mileage constraint.

$$Z = 400(S_{t1}+S_{c1}) + 600(S_{t2}+S_{c2}) + 150(i_{t1}+i_{t2}+i_{c1}+i_{c2})$$

Constraints:

(Inventory Requirements)

$$i_{t1} = i_{t0} + X_{t1} - 400$$
$$i_{t2} = i_{t1} + X_{t2} - 300$$
$$i_{c1} = i_{c0} + X_{c1} - 800$$
$$i_{c2} = i_{c1} + X_{c1} - 300$$

(Max Production)

$$X_{t1} + X_{c1} \le 1000$$
$$X_{t2} + X_{c2} \le 1000$$

(Max Steel)

$$S_{t1} + S_{c1} \le 1500$$
$$S_{t2} + S_{c2} \le 1500$$

(Steel to Car Requirements)

$$2X_{t1} \le S_{t1}$$
$$2X_{t2} \le S_{t2}$$
$$X_{c1} \le S_{c1}$$
$$X_{c2} \ge S_{c2}$$

(mpg requirements)

$$\frac{22X_{t1} + 36X_{c1}}{X_{t1}+X_{c1}} \ge 30$$
$$\frac{22X_{t2} + 36X_{c2}}{X_{t2}+X_{c2}} \ge 30$$

$\Rightarrow$

$$\begin{array}{l}
X_{t1} - i_{t1} = 300 \\
X_{t2} + i_{t1} - i_{t2} = 300 \\
X_{c1} - i_{c1} = 600 \\
X_{c2} + i_{c1} - i_{c2} = 300 \\
X_{t1} + X_{c1} \le 1000 \\
X_{t2} + X_{c2} \le 1000 \\
S_{t1} + S_{c1} \le 1500 \\
S_{t2} + S_{c2} \le 1500 \\
2X_{t1} - S_{t1} \le 0 \\
2X_{t2} - S_{t2} \le 0 \\
X_{c1} - S_{c1} \le 0 \\
X_{c2} - S_{c2} \le 0 \\
-8X_{t1} + 6X_{c1} \ge 0 \\
-8X_{t2} + 6X_{c2} \ge 0
\end{array}$$

Solution: From Excel:

$X_{t1}=400$, $X_{t2}=200$, $X_{c1}=600$, $X_{c2}=300$

$i_{t1}=100$, $i_{t2}=i_{c1}=i_{c2}=0$

$S_{t1}=800$, $S_{t2}=400$, $S_{c1}=600$, $S_{c2}=300$

Min Cost $Z = \$995,000$