Section 4.5: The Simplex Algorithm

- Step-by-Step description of the Simplex Algorithm.

  1.) Convert the LP to standard form.

  2.) Obtain a bfs (if possible) from the standard form.*

  3.) Determine whether the current bfs is optimal.

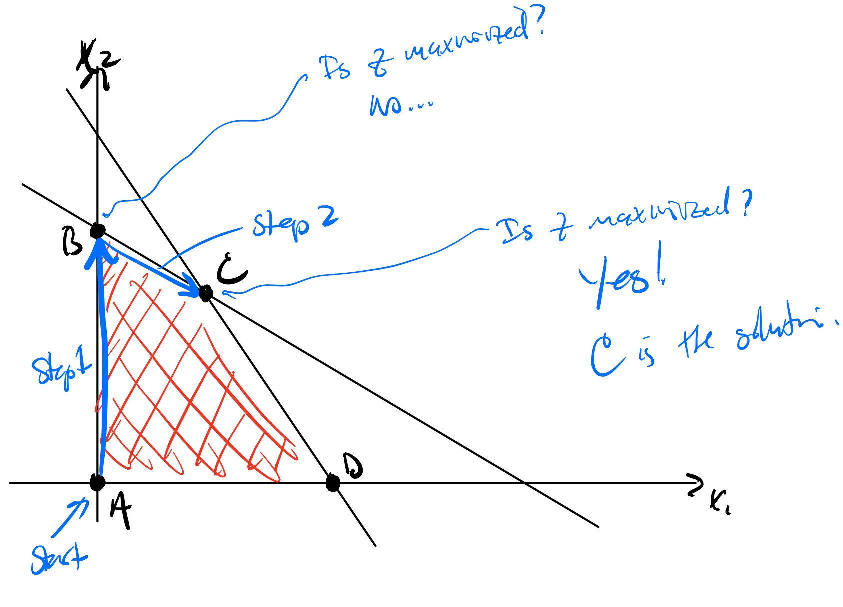
  4.) ~~If~~ the current bfs is not optimal, then determine which nonbasic variable to find a new bfs with a better objective function value.

  5.) Use EROs to find new bfs with better obj. fun. value, then repeat steps 3–5 if necessary.

* Typically start with the origin.

- The Simplex Algorithm is an iterative method, and so the computations involved are repetitious. They are also very tedious.

- Graphically, the Simplex Method will search along the boundary of the feasible space by identifying the corner point (i.e. bfs) with the largest obj. fun. value.



- Although you will typically never compute a solution to an LP by hand, it is good to understand how it works by experiencing it by hand.

$\boxed{\text{WS}}$ #1 Solving the Reddy Mikks Problem using Simplex Algorithm

- We can summarize the process of the Simplex Algorithm as follows:

1.) Optimality Condition: The entering variable in a maximization problem is the nonbasic variable with the most negative coefficient in the z-row. Ties are broken arbitrarily. The optimum is reached at the iteration where all the z-row coefficients are non-negative.

2.) Feasibility Condition: The leaving variable is the basic variable associated with the smallest non-negative ratio with strictly positive denominator. Ties are broken arbitrarily.

3.) Gauss-Jordan Row Ops:

a.) Pivot Row:

→ Replace leaving var. in the Basic Column with entering var.

→ New Pivot Row = Current Pivot Row ÷ Pivot Element

b.) All other rows, including z:

→ New Row = Current Row − (Pivot Column Coeff) x (New Pivot Row)

$\boxed{\text{WS}}$ #1-#2 working with solving maximization LPs with Simplex Algorithm

• Quick note: when all constraints are "≤" constraints, the initial bfs can easily be the origin (i.e. all slack variables are basic variables.) Otherwise, the initial bfs is not readily apparent.

Section 4.6: Simplex Algorithm for Minimization LPs.

• To solve an LP with a minimization goal, we can modify our original method in two ways:

1.) Change z to $-z$ and maximize $-z$.

Ex: obj. fun.: Minimize $z = 2x_1 - 3x_2$.

Change to

obj. fun: Maximize $-z = -2x_1 + 3x_2$

The only difference is then to modify Row 0 in the Simplex Tableau

Row 0 | $-z$ | 1 -2 3   0 . . .

and solve in the normal way.

2.) Entering variables correspond to the most positive element in Row 0. Proceed as normal but change the criteria for an entering variable. The tableau is optimal when all coefficients in Row 0 are negative.

WS

Section 4.7: Alternative Optimal Solutions

- Recall from graphical two-variable problems that an LP may have an infinite number of optimal solutions if the objective function creates an isocost/isoprofit line that is parallel to a non-redundant constraint.

- In terms of the feasible region, this means any point on the edge connecting two optimal corner points is also optimal. How does the Simplex Tableau represent an LP with alternative optima?

[WS] #1 work with an LP with alternative optima.

- In short, an LP will yield alternative optima if there exists a non-basic variable with a zero coefficient in Row 0.

Section 4.8: Unbounded LPs

- We demonstrate the unboundedness of a maximization LP by considering an example.

[WS] #1 Apply Simplex Algorithm to an unbounded LP.

- From the example in the worksheet we see that an unbounded LP will occur in a maximization problem if the following occurs:

1.) A nonbasic variable has a negative coefficient in Row 0 i.e. introducing a nonbasic variable into the basis will increase the objective function value.

2.) There is no constraint that limits how large we can make this entering variable.

- The second point above is problem dependent; however, for an LP with "$\leq$" constraints, this means the entering variable column has non-positive entries.

- Hence, for a maximization LP, if the tableau has a column with a negative coefficient in Row 0 with non-positive coefficients in all constraint rows, the problem is unbounded.

- Another characteristic that defines unboundedness is if there exists a direction of unboundedness, $\vec{d}$, such that $\vec{c}^T\vec{d} > 0$, where $\vec{c}$ is the column vector of objective function constraints. This holds for maximization LPs.

- For minimization LPs with "$\leq$" constraints, unboundedness occurs when
  - there exists an entering variable column with a positive coefficient in Row 0 with non-positive coefficients in the constraint rows.
  - there exists a direction of unboundedness, $\vec{d}$, such that $\vec{c}^T\vec{d} < 0$.

$\boxed{\text{WS}}$ #2-#3 working with unbounded LPs.

Section 4.11: Degeneracy & Convergence of the Simplex Method

Def: An LP is degenerate if it has at least one basic feasible solution in which a basic variable is 0.

- The Simplex Method is an iterative method, and characteristic of some iterative methods is the idea of cycling (e.g. remember Newton's method?)

• Recall that to enter a new variable into the basis, we must (at some point) compute the new value of the objective function. This calculation is, for example,

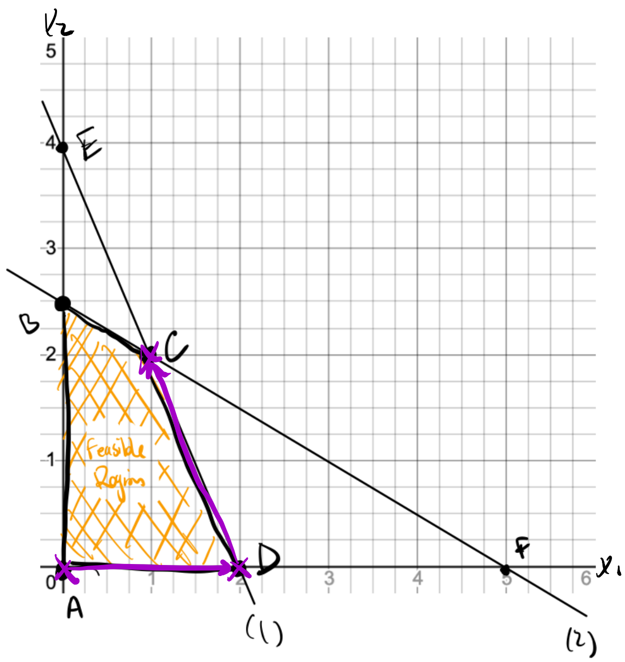
This # is typically the RHS value of Pivot Row

$$[\text{New } z] = [\text{Old } z] - [\text{value of entering variable}] \times [\text{Coeff in Row } 0]$$

what if this is zero?

We see that if the entering variable is 0, the updated z-value doesn't change i.e. we've wasted an iteration — our new bfs is essentially the same as the last bfs.

Ex: Consider LP from Section 4.2:



Maximize: $z = 2x_1 + 3x_2$

Subject to:
$$2x_1 + x_2 \le 4$$
$$x_1 + 2x_2 \le 5$$
$$x_1, x_2 \ge 0$$

Standard form:

$$2x_1 + x_2 + s_1 = 4$$
$$x_1 + 2x_2 + s_2 = 5$$

$n=4$, $m=2$ $\to$ ${}_nC_m = {}_4C_2 = 6$

• There are 6 basic solutions.
• They correspond to the 6 intersections.

	Nonbasic Variables	Basic Variables	Basic Solution	Corner Point	Feasible?	Obj Value (z)
Iteration 0	x_1, x_2	s_1, s_2	$s_1=4, s_2=5$	A	Yes	$z=0$
	x_1, s_1	x_2, s_2	$x_2=4, s_2=-3$	E	No	—
	x_1, s_2	x_2, s_1	$x_2=5/2, s_1=3/2$	B	Yes	$z=15/2$
Iteration 1	x_2, s_1	x_1, s_2	$x_1=2, s_2=3$	D	Yes	$z=4$
	x_2, s_2	x_1, s_1	$x_1=5, s_1=-6$	F	No	—
Iteration 2*	s_1, s_2	x_1, x_2	$x_1=1, x_2=2$	C	Yes	$z=8$ OPTIMAL

- This problem is nondegenerate since every bfs has a non-zero basic variable value. Cycling cannot occur and every new bfs yields a better obj. fun. value as the algorithm progresses.

- Now, consider a problem that is degenerate. The difference can be seen in the correspondence of basic solutions to the corner points of the feasible space.

WS #1 Graphical significance of degeneracy.

- Degeneracy is typically brought on by redundant constraints and/or constraints whose graphical representations intersect at the same corner point.
- In the Simplex Tableau, a degeneracy usually occurs when the RHS of a constraint is 0 or when a tie occurs in the ratio test.

- To avoid cycling, Bland's Rule may be applied. It can be shown that the following criteria for selecting the entering/leaving variable in the presence of ties will always break a cycle:

  1.) In a max LP with n decision variables, label all slack/excess variables as $x_{n+1}, x_{n+2}, \ldots$

  2.) Choose the entering variable (in a max problem) as the variable with a negative coefficient in Row 0 with the smallest subscript.

  3.) Break a ratio test tie by choosing the winner to be the leaving variable with the smallest subscript.

WS #1 Demonstrate cycling, then break it using Bland's Rule.

- Although Bland's Rule works in theory, it is rarely implemented due to high computational costs. Also, in practice, cycling is rare.

Section 4.12: The Big M Method.

- Up until now, we've only considered max or min LPs with "≤" constraints.
- The Big M Method and the Two-Phase Method (Section 4.13) are ways to initiate the Simplex Method for an LP with "=" or "≥" constraints.

- The Big M Method (briefly):
  - Eliminate any negative coefficients on the RHS of any constraints by multiplying by -1 and appropriately "flipping" the inequalities.
  - Cast the LP into standard form by introducing slack and excess variables to "≤" and "≥" constraints, respectively.
  - Suppose equation i does not have a slack variable (i.e. it was initially an equality constraint or it has an excess variable), then add an artificial variable, a_i, to the LHS of the constraint (not unlike adding a slack variable). This allows us to have a starting bfs.

Ex: Suppose we have the following constraints:

$$\begin{aligned} 4x_1 + 3x_2 - 4x_3 &\geq 6 \\ 3x_1 - 4x_2 &\geq -5 \\ 2x_1 + 3x_3 &= 1 \end{aligned}$$

1.) First eliminate all negative RHS:

$$\Rightarrow \begin{aligned} 4x_1 + 3x_2 - 4x_3 &\geq 6 \\ -3x_1 + 4x_2 &\leq 5 \\ 2x_1 + 3x_3 &= 1 \end{aligned}$$

2.) Cast into Standard Form:

$$\Rightarrow \begin{aligned} 4x_1 + 3x_2 - 4x_3 - e_1 &= 6 \\ -3x_1 + 4x_2 + S_2 &= 5 \\ 2x_1 + 3x_3 &= 1 \end{aligned}$$

3.) Add artificial variables to LHS of equations without slack variables:

$$\Rightarrow \begin{aligned} 4x_1 + 3x_2 - 4x_3 - e_1 + a_1 &= 6 \\ -3x_1 + 4x_2 + S_2 &= 5 \\ 2x_1 + 3x_3 + a_3 &= 1 \end{aligned}$$

- We've introduced new artificial variables, a_i. An optimal solution to this new system will be an optimal solution to the original LP only if $a_i = 0$. How do we ensure this? We introduce a penalty for these variables in the objective function.

Ex: Suppose we have the following objective function:

$$\text{Maximize } Z = 3x_1 + 2x_2 + x_3$$

(Introduce a_1 and a_3 into constraints as necessary)

Add penalty: $\text{Maximize } Z = 3x_1 + 2x_2 + x_3 - Ma_1 - Ma_3$

where M is an arbitrarily large number ($M \to \infty$). For a minimization problem, we would add "Ma_1" and Ma_3:

Add penalty: $\text{Minimize: } Z = 3x_1 + 2x_2 + x_3 + Ma_1 + Ma_3$

- From here, we construct an initial tableau after eliminating the artificial variables from the objective function using the constraints. This is because when we initially enter a_1 and a_2 into the basis, their Row 0 values are non-zero (Recall, to have a tableau in standard form all basic variables have a coefficient of 0 in Row 0.) We illustrate this with an example.

WS #1 work on constructing the initial tableau.

- Once we establish the initial tableau, we use the Simplex (along with any other techniques) to find an optimal solution.

WS #2 solve an LP using the Big M Method.

- If any artificial variables, a_i, are included in the optimal set of basic variables with $a_i > 0$ for some i, then the original LP is infeasible.

Section 4.13: The Two-Phase Simplex Method

- In the Big M Method, the use of the coefficient M as a penalty may lead to computer round-off error.
- The Two-Phase Method eliminates the use of M altogether.

Phase I: Introduce artificial variables into the constraints as in the Big M Method. Define w as a new objective function as the sum of all artificial variables. Regardless if the LP is a maximization/minimization problem, solve this new problem as a minimization LP. If the optimal solution is positive, the original LP is infeasible. Otherwise, min $w = 0$ and proceed to Phase II.

Phase II: Use the feasible solution from Phase I as a starting basic feasible solution for the original problem.

Note: When starting each phase, we may have to eliminate basic variables from Row 0 as in the Big M Method.

When doing the Two-Phase Method three cases may arise when transitioning from Phase I to Phase II.

- Case I: worst case scenario - the optimal value of w in phase I is positive. This means the original LP is infeasible.

- Case II: Best case scenario - the optimal value of w is zero and all artificial variables are zero. Proceed to Phase II as normal.

- Case III: Meh scenario - the optimal value of w is zero and all artificial variables are zero but at least one artificial variables is in the optimal basis. Two additional steps are required.

Step 1: Select an artificial variable in the basis to leave the basis and designate its row as the pivot row. The entering variable can be any nonbasic original variable with a nonzero Row 0 coefficient. Perform an iterations.

Step 2: Remove the column of just-leaving artificial variable. If all artificial variables have been removed, go to Phase II. Otherwise, repeat step 1.

$\boxed{\text{WS}}^{P1}$ #1 working with The Two Phase Method in Case II.

$\boxed{\text{WS}}^{P2}$ #1 working with The Two Phase Method in Case III