MAT 361 - Operations Research
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CHAPTER 4 - Simplex Algorithm

Section 4.1: Converting an LP to Standard Form

- An LP may have both equality or inequality constraints, and typically the variables are non-negative. SJ

- Def: The standard form of an LP is an equivalent problem formulation of that LP with only equality constraints and non-negative requirements on all variables.

- To successfully convert an LP to standard form, we introduce slack and/or excess variables.

1) Slack: consider the constraint

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$

Introduce s_1, slack variable for constraint 1:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + s_1 = b.$$

Note: s_1 is the amount needed (slack) to make the inequality constraint binding. Also, $s_1 \geq 0$.

2.) Excess: consider the constraint

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b$$

Introduce e_1, excess variable for constraint 1:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n - e_1 = b$$

Note: e_1 is the overflow amount (excess). Subtracting this amount makes the inequality constraint binding. Also, $e_1 \geq 0$.

- Once the constraints are made into equalities, we can write the LP in matrix-vector notation.
- Suppose our standard form has n decision variables $(x_1, x_2, \ldots, x_n)$, which may include slack/excess variables. Assume there are m constraints. Then

$$\text{Max (or Min) } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + c_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + c_{2n}x_n &= b_2 \\ \vdots \qquad\quad \ddots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + c_{mn}x_n &= b_m \end{aligned}$$

$$x_i \geq 0$$

Defining $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ as the matrix of constraint coefficients, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ as the vector of constraint requirements, and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ as the vector of decision variables, we have

LP in Standard Form:

$$\boxed{\begin{aligned} &\text{Max (or Min)} \quad z = \sum_{i=1}^{n} c_i x_i \\ &\text{Subject to:} \quad A\vec{x} = \vec{b} \\ &\qquad\qquad\qquad x_i \geq 0 \end{aligned}}$$

- Note: A is $m \times n$, $\vec{b}$ is $m \times 1$, and $\vec{x}$ is $n \times 1$. In text, vectors are denoted by bold print.
- Note: we could define $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ to be the optimal solution vector coefficients and write the obj function as

$$z = \vec{c}^T \vec{x}$$

$\boxed{\text{WS}}$ #1–#2 working with converting an LP to Standard Form.

Section 4.2: A Preview of the Simplex Method.

- In standard form, the system $A\vec{x} = \vec{b}$ represents the feasible region.

- Note that for all non-trivial LP problems, the number of constraints (m) is always less than the number of decision variables (n), i.e. $n > m$.

Ex: ($n = m$) $\quad x = 1 \quad n=1,\ m=1 \longrightarrow$ Unique Solution: $x=1$

* Ex: ($n > m$) $\quad x+y=1 \quad n=2,\ m=1 \longrightarrow$ Infinite # of Sols

Ex: ($n < m$) $\quad 2x=2,\ x=1 \quad n=1,\ m=2 \longrightarrow$ Unique Solution: $x=1$

- In the third example above, we see that one of the equations is redundant. Also, in each equation above, the equations given are consistent (at least one solution). In the case of $n<m$, we'll always have a redundant equation.

- In general, we navigate the feasible region of $A\vec{x}=\vec{b}$ by first identifying Basic and Nonbasic Variables.

- Def: In system of n variables and m equations with $n>m$, the set of nonbasic variables is a set of $n-m$ variables set equal to zero. The remaining m variables are the basic variables. Solving the m equations in the remaining m basic variables yields a basic solution.

- Let A be an $m \times n$ matrix with $n>m$, then in the equation $A\vec{x}=\vec{b}$, a set of nonbasic and basic variables may take the following form:

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} & a_{m,m+1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \left.\begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{matrix}\right\}\text{Basic} \\ \left.\begin{matrix} x_{m+1} \\ \vdots \\ x_n \end{matrix}\right\}\text{Non Basic set to 0} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} & a_{m,m+1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \left.\begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{matrix}\right\}\text{Basic} \\ \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\}\text{Non Basic set to 0} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Useless

$$\Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \longrightarrow$$ Solving this system yields a basic solution.

- A few things to note:
  - The nonbasic variables can be any subset of $n-m$ variables
  - The nonbasic variables are set to zero.
  - Solving the resulting $m \times m$ system may not yield a solution!
  - A basic solution may or may not be feasible!
  - In a 2-variable problem, a basic solution is the intersection of two lines!
  - Since we have n variables and a subset of those variables of size m forms a basic solution, there are potentially n choose m, ${}_nC_m$, basic solutions

$${}_nC_m = \frac{n!}{m!(n-m)!}$$

- <u>Def</u>: A basic solution in which all variables are non-negative is called a <u>basic feasible solution</u>.

- Basic Roadmap:

Two-Var Problems → **Graphical Method:**

1.) Graph constraints, identify feasible region, infinite # of possible solutions.

↓

2.) Identify feasible corner points; candidates for optimal solution are reduced to a finite set.

↓

3.) Use Obj. Fun. to determine the optimal corner point.

Algebraic Method:

1.) Represent solution space by m eqns in n variables, $n > m$; system has an infinite # of feasible sols.

↓

2.) Identify the feasible basic solutions; candidates for optimal solution are reduced to a finite set.

↓

3.) Use obj. fun. to determine the optimal feasible basic solution.

$\boxed{\text{WS}}$ #1-#2 working with basic solutions.

- we note a few things from the examples in the previous worksheet:
  - As the size of the LP problem increases, enumerating the corner points becomes a daunting task e.g. if $m=10$ and $n=20$,

$${}_{20}C_{10} = \frac{20!}{10!\,10!} = 184{,}756$$

and $m=10, n=20$ is relatively small.

- As we'll see, it turns out that a point in the feasible region is an extreme (corner) point if and only if it is a basic feasible solution to the LP.
- Further, if an LP has an optimal solution, then it has an optimal basic feasible solution.
- The simplex algorithm is an iterative method that investigates only a "select few" of these basic feasible solutions.

Section 4.3: Direction of Unboundedness

- Recall that a convex set is a set for which the line connecting any two points in that set is wholly contained in that set.
- Recall that the feasible region of an LP is convex. This means that if $\vec{x}_1, \vec{x}_2 \in S$, a convex set, then $\vec{y} = \sigma_1 \vec{x}_1 + \sigma_2 \vec{x}_2$ is also in the set S, where $\sigma_1 + \sigma_2 = 1$.

$\boxed{\text{WS}}$ #1, #2 working with a quick example.

- For an n-dimensional convex set, any linear combination of n vectors of the form

$$\vec{y} = \sigma_1 \vec{x}_1 + \sigma_2 \vec{x}_2 + \cdots + \sigma_n \vec{x}_n = \sum_{i=1}^{n} \sigma_i \vec{x}_i$$

is in S provided $\sum_{i=1}^{n} \sigma_i = 1$, $\sigma_i \geq 0$. This is called a <u>convex combination</u>.

- For any LP with an unbounded feasible region, there exists a vector, $\vec{d}$, for which if we move in the direction of that vector we will remain within the feasible space.
- Denote $\vec{0}$ as the origin or vector of zeros, $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.
- Def: Suppose the system $A\vec{x} = \vec{b}$ represents the feasible space, S, an LP, where $\vec{x}$ is any feasible solution with $\vec{x} \geq \vec{0}$. Then a nonzero vector $\vec{d}$ is a direction of unboundedness if for all $\vec{x} \in S$ and any scalar $c > 0$, the vector $\vec{y} = \vec{x} + c\vec{d}$ is also in S.

- In short, $\vec{d}$ is a direction for which we can move as far as we want and still be in the feasible region.

Thm: Consider an LP in standard form, having basic feasible solutions given by $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k$. Let $\vec{x}$ be any point in the LP's feasible region, then $\vec{x}$ can be written as

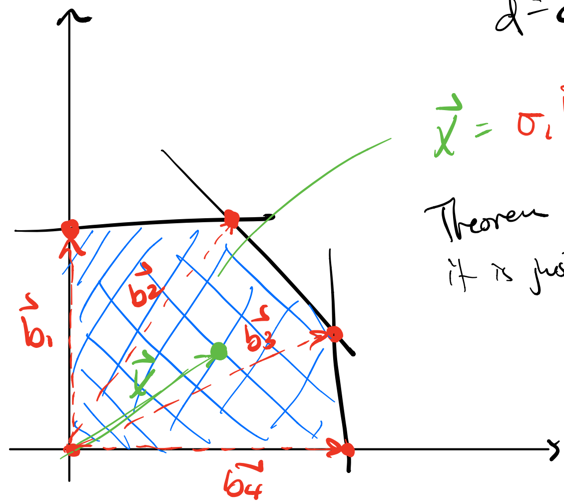
$$\vec{x} = \vec{d} + \sum_{i=1}^{k} \sigma_i \vec{b}_i$$

where $\vec{d} = \vec{0}$ or $\vec{d}$ is a direction of unboundedness and

$$\sum_{i=1}^{k} \sigma_i = 1, \quad \sigma_i \geq 0.$$

- For a bounded feasible region: $\vec{d} = \vec{0}$ and this theorem simply states that any feasible point can be written as a convex combination of the basic feasible solutions.

Ex:



$\vec{d} = \vec{0}$

$\vec{x} = \sigma_1 \vec{b_1} + \sigma_2 \vec{b_2} + \sigma_3 \vec{b_3} + \sigma_4 \vec{b_4}$

Theorem guarantees this is possible, but it is just the definition of a convex combination.

- For an unbounded feasible region: $\vec{d} \neq \vec{0}$, and we must include the vector $\vec{d}$ in the convex combination to obtain any arbitrary vector in the feasible space. How do we determine $\vec{d}$?

$\boxed{\text{WS}}$ #3 An example to show $\vec{d}$ does exist.

- Thm: For an LP with an unbounded feasible region, $\vec{d}$ is a direction of unboundedness if and only if $A\vec{d} = \vec{0}$ and $\vec{d} \geq 0$.

Proof: ($\Rightarrow$) Suppose $\vec{d}$ is a direction of unboundedness, then for all $x \in S$, S being the feasible space, there exists a vector $\vec{y} = \vec{x} + c\vec{d}$, with $c > 0$, such that $\vec{y} \in S$. That is, $\vec{x}$ and $\vec{y}$ both satisfy $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$ as they are both feasible solutions. We have two cases:

If $\vec{d} = 0$: Then $A\vec{d} = \vec{0}$.

If $\vec{d} > 0$: Then $\vec{d} = \frac{1}{c}\vec{y} - \frac{1}{c}\vec{x}$, which gives

$$A\vec{d} = A\left(\frac{1}{c}\vec{y} - \frac{1}{c}\vec{x}\right) = \frac{1}{c}A\vec{y} - \frac{1}{c}A\vec{x} = \frac{1}{c}\vec{b} - \frac{1}{c}\vec{b} = \vec{0},$$

as desired.

($\Leftarrow$) Suppose $A\vec{d}=\vec{0}$ with $\vec{d}\geq 0$, then we wish to show the existence of the feasible solution $\vec{y}=\vec{x}+c\vec{d}$ i.e. we need to show $A\vec{y}=\vec{b}$ assuming $A\vec{x}=\vec{b}$. We have

$$A\vec{y}=A(\vec{x}+c\vec{d})=A\vec{x}+cA\vec{d}=\vec{b}+c(\vec{0})=\vec{b},$$

as desired.

- This theorem says that the direction of unboundedness resides in the <u>null space</u> of the matrix A i.e. all vectors $\vec{x}$ such that $A\vec{x}=\vec{0}$.

$\boxed{\text{WS}}$ #4 Finding the null space of a matrix.

<u>Section 4.4</u>: Why Does an LP Have an Optimal Basic Feasible Solution?

- The Simplex Algorithm, which we have by now over-hyped, hinges on the following theorem (that will go without proof).
- <u>Thm</u>: A point in the feasible region of an LP is an extreme point (corner point) if and only if it is a basic feasible solution of the LP.

- In this section, we show that if the LP has an optimal solution, then it has an optimal basic feasible solution.

- First, we cast the LP into the following form:

$$\begin{aligned} &\text{Maximize (or Minimize)} \quad z=\vec{c}^T\vec{x} \\ &\qquad\qquad \text{Subject to:} \quad A\vec{x}=\vec{b} \\ &\qquad\qquad\qquad\qquad\quad \vec{x}\geq\vec{0} \end{aligned}$$

where $\vec{c}=\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is the vector of coefficients in the objective function.

Thm: If an LP has an optimal solution, then it has an optimal basic feasible solution.

Proof: Suppose $\vec{x}$ is an optimal solution to the LP above and suppose $z_{opt} = \vec{c}^T\vec{x}$ is the optimal value of the objective function. Since $\vec{x}$ is feasible, $\vec{x}$ can be written as

$$\vec{x} = \vec{d} + \sum_{i=1}^{k} \sigma_i \vec{b_i}$$

where $\vec{d} = \vec{0}$ or $\vec{d}$ is a direction of unboundedness, $\vec{b_i}$ are the basic feasible solutions ($k \leq {}_nC_m$), and $\sum_{i=1}^{k} \sigma_i = 1$ with $\sigma_i \geq 0$.

Consider the dot product $\vec{c}^T\vec{d}$. If $\vec{c}^T\vec{d} > 0$, then for any $k > 1$, the feasible solution

$$\vec{y} = k\vec{d} + \sum_{i=1}^{k} \sigma_i \vec{b_i}$$

yields an optimal value of z greater than z_{opt} because

$$\begin{aligned}
\vec{c}^T\vec{y} &= \vec{c}^T\left(k\vec{d} + \textstyle\sum_i \sigma_i \vec{b_i}\right) \\
&= k\vec{c}\vec{d} + \vec{c}\textstyle\sum_i \sigma_i \vec{b_i} \\
&> \vec{c}^T\vec{d} + \vec{c}\textstyle\sum_i \sigma_i \vec{b_i} \qquad \text{) } \vec{c}^T\vec{d} > 0 \\
&= \vec{c}^T\left(\vec{d} + \textstyle\sum_i \sigma_i \vec{b_i}\right) \\
&= \vec{c}^T\vec{x} \\
&= z_{opt}
\end{aligned}$$

This contradicts the optimality of $\vec{x}$. In fact as k grows, the solution $\vec{y}$ is still feasible and the dot product $\vec{c}^T\vec{y}$ grows without bound suggesting the LP does not have an optimal solution. This is a contradiction and so $\vec{c}^T\vec{d} \not> 0$.

If $\vec{c}^T\vec{d} < 0$, then the feasible point $\vec{z} = \sum_{i=1}^{k} \sigma_i \vec{b_i}$ has a larger optimal value than z_{opt} because

$$\vec{c}^T\vec{z} = \vec{c}^T \textstyle\sum_i \sigma_i \vec{b_i}$$
$$> \vec{c}^T\vec{d} + \vec{c}^T \textstyle\sum_i \sigma_i \vec{b_i} \quad \Big) \ \vec{c}^T\vec{d} < 0$$
$$= \vec{c}^T(\vec{d} + \vec{c}^T \textstyle\sum_i \sigma_i \vec{b_i})$$
$$= \vec{c}^T\vec{x}$$
$$= z_{opt}.$$

This again contradicts the optimality of $\vec{x}$ and so we cannot have $\vec{c}^T\vec{d} < 0$. Thus, we have $\vec{c}^T\vec{d} = 0$, if $\vec{x}$ is optimal. This means

$$z_{opt} = \vec{c}^T\vec{x}$$
$$= \vec{c}^T(\vec{d} + \textstyle\sum_i \sigma_i \vec{b_i})$$
$$= \vec{c}^T \textstyle\sum_i \sigma_i \vec{b_i}$$
$$\Rightarrow \quad z_{opt} = \sum_{i=1}^{k} \sigma_i \vec{c}^T \vec{b_i}$$

Now suppose $\vec{b_n}$ is the basic feasible solution with the largest objective function value, i.e. $\vec{c}^T\vec{b_n} \geq \vec{c}^T\vec{b_i}$ for all $i \neq n$.
Then

$$\vec{c}^T\vec{b_n} = \vec{c}^T\vec{b_n}\Big(\overbrace{\sum_{i=1}^{k}\sigma_i}^{=1}\Big)$$
$$= \sum_{i=1}^{k} \sigma_i \vec{c}^T\vec{b_n}$$
$$\geq \sum_{i=1}^{k} \sigma_i \vec{c}^T\vec{b_i}$$
$$= z_{opt}.$$

Therefore, it follows that $\vec{b_n}$ is also optimal and the LP does indeed have an optimal basic feasible solution. ∎

- We essentially now have the green light to only check the set basic feasible solutions for optimality — enter the Simplex Algorithm.
- One last definition:

Def: For any LP with m constraints, two feasible solutions are said to be adjacent if their sets of basic variables have $m-1$ basic variables in common.