MAT 361 – Operations Research
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CHAPTER 3 – Linear Programming

Section 3.1: Elements of LP

- An LP-problem is an optimization problem for which the objective function and constraints are linear.

- Def: A function of n variables, $f(\vec{x})$ or $f(x_1, x_2, \ldots, x_n)$ is a linear function if and only if for some set of constants $c_1, c_2, c_3, \ldots, c_n$,

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Note: this function passes through origin $\vec{0} = (0,0,0,\ldots,0)$.
Note: f could also be written

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} c_kx_k$$

or for $\vec{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$

$$f(\vec{x}) = \vec{c} \cdot \vec{x} = \vec{c}\,\vec{x}^T$$

- Def: For any linear function $f(x_1, x_2, \ldots, x_n)$ and any # b, the following are linear inequalities

$$f(x_1, x_2, \ldots, x_n) \leq b \qquad f(x_1, x_2, \ldots, x_n) \geq b.$$

• **Def:** An LP-problem is an optimization problem for which we do the following:

1.) Max (or Min) a linear function of decision variables. This function is called the objective function.

2.) The values of the decision variables satisfy a set of constraints, which are linear equations or inequalities.

3.) A sign restriction is associated with each decision variable.

$\boxed{\text{WS}}$ #1 The Reddy Mikks Problem

• From the worksheet on the Reddy Mikks Problem we obtain the following LP:

$$\boxed{\begin{aligned} &\text{Max } z = 5x_1 + 4x_2 \\ &\text{s.t.} \\ &\quad 6x_1 + 4x_2 \le 24 \\ &\quad x_1 + 2x_2 \le 6 \\ &\quad -x_1 + x_2 \le 1 \\ &\quad x_2 \le 2 \\ &\quad x_1, x_2 \ge 0 \end{aligned}}$$

LP-Problem

• Terminology (some from Chapter 1):

- There are two decision variables, x_1 & x_2, which makes this a 2-Dimensional or 2-variable LP.

- The objective function, $z = 5x_1 + 4x_2$, is linear.
- The constraints are linear inequalities.

Immediately Satisfies Proportionality & Additivity Assumptions

- The variable x_1 & x_2 are taken to be real #'s, so this is a non-integer model.
  - In some cases, we may be "ok" with fractional output even though it may not be physically possible — Divisibility Assumption (may need to use rounding, but be careful)
  - Some LPs are integer models where at least one decision var must be integer valued (more difficult)
- We are assuming this model is deterministic, which means the objective function values and the constraint inequalities are known with certainty for any values of x_1 & x_2

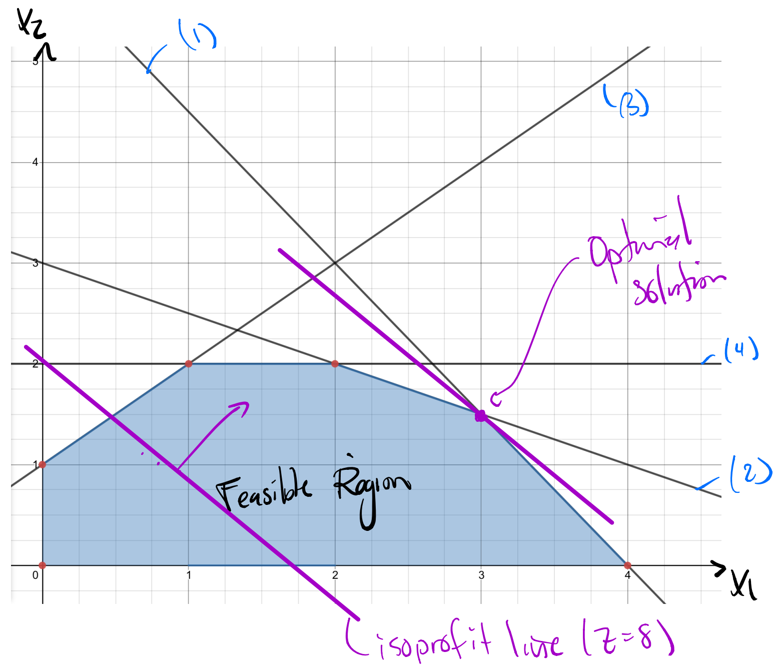
  Ex: Reddy Mikks problem: we assume that it requires exactly 6 tons of M_1 to produce a ton of x_1. For many realistic processes, this value "6" may be more variable. So, in a sense, deterministic is based on average values.

  - For many human processes, variables may carry along a variability/variance. Building a model that incorporates such random factors is called stochastic.
  - Ignoring the fact that variability is involved in any manufacturing process is essentially the Certainty Assumption.
- This is a static model as we're making a decision for a single time period (i.e. now).
  - A dynamic model may be used to predict or make a decision for multiple time periods in the future.

- To solve this LP would mean to find the optimal solution.
  - Def: A feasible solution is a point that satisfies all constraints and sign restrictions. The set of all feasible solutions is called the feasible region.
  - Def: The optimal solution for a maximization (minimization) problem is a point in the feasible region that yields the largest (smallest) value of the objective function.

Section 3.2: Graphical Solution to Two-Var LP Problems

- Consider the feasible region obtained from the Reddy Mikks problem:



LP-Problem

$$\text{Max } z = 5x_1 + 4x_2$$

s.t.

$$6x_1 + 4x_2 \le 24 \quad (1)$$
$$x_1 + 2x_2 \le 6 \quad (2)$$
$$-x_1 + x_2 \le 1 \quad (3)$$
$$x_2 \le 2 \quad (4)$$
$$x_1, x_2 \ge 0$$

- To solve this problem graphically, we first identify the feasible region.
- Then construct the isoprofit line (or isocost line for min problems).

Pick a value for z in the objective function

$$z = 8 \Rightarrow 8 = 5x_1 + 4x_2 \Rightarrow \boxed{x_2 = -\tfrac{5}{4}x_1 + 2}$$ Isoprofit Line

- Note that for different values of z, the isoprofit line has the same slope. Thus, as z increases, each new line is parallel to the previous. Identify the <u>direction of increase</u> as we change the value of z.
- Note that if the isoprofit line intersects the feasible region, then these points in the feasible region on the line will produce that value of the objective function.

Ex: $\underbrace{X_1 = 2,\ X_2 = 1}_{\text{In feasible region.}}$ $\underbrace{z = 5(2) + 4(1) = 14}_{\text{profit.}}$

Thus, (2,1) is in the feasible region and would lie on the isoprofit line

$$14 = 5x_1 + 4x_2$$

$$\Rightarrow \quad X_2 = -\frac{5}{4}X_1 + 7$$

- We continue to move the isoprofit line in the direction of increasing z (northeast) until the last point in the feasible region still lies on the isoprofit line. This single point is the <u>optimal solution</u>:
  - The optimal solution is at $(X_1, X_2) = (3, 1.5)$, which is a <u>corner point</u> on the feasible region.

$\boxed{\text{WS}}$ #2–#4 Finish off the Reddy Mikks Problem.

• A 3-dimensional interpretation of the Two-variable LP-Problem: Consider the Reddy Mikks problem again.

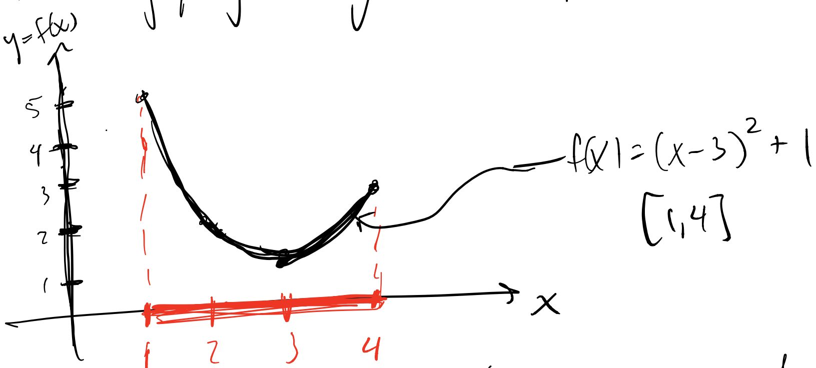
$$\text{Max } z = 5x_1 + 4x_2$$

S.T.

$$\left.\begin{aligned} 6x_1 + 4x_2 &\le 24 \\ x_1 + 2x_2 &\le 6 \\ -x_1 + x_2 &\le 1 \\ x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}\right\}$$

Constraints form the feasible Region.

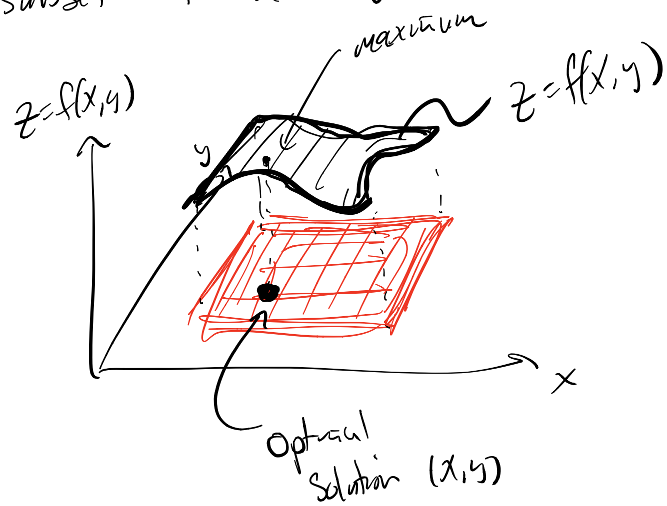
• Recall when graphing a single variable function on a restricted domain:



• The black curve is essentially the image of the portion of the number line between $x=1$ and $x=4$ (the red line) produced by the function f.

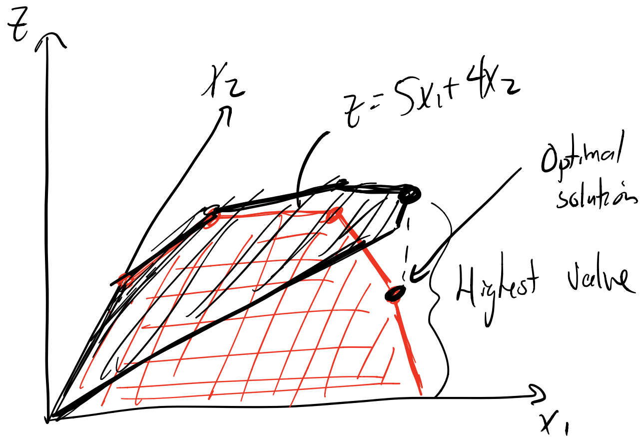
• The height of the curve is the function value. On [1,4], it attains a maximum at $x=1$.

• Likewise, consider a two-dimensional function $z=f(x,y)$ applied to a subset of the x-y plane:

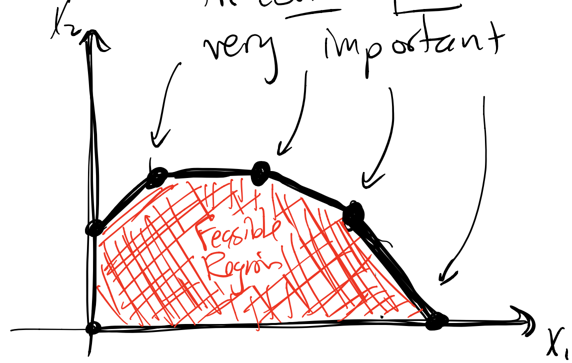


• The value inside the red plane that maximizes the height of the surface is the optimal solution.

- In the Reddy Mikks problem (or a typical LP-Problem), the function z is a plane through the origin and the feasible space is this red subset of the x-y plane



- Because z is a plane, the corner points are very important



WS #1-#2 Formulating LP-Problems

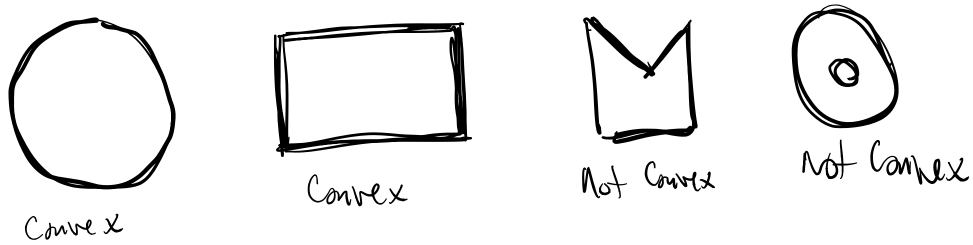
- A few definitions:
  - A constraint is <u>redundant</u> if deleting it doesn't change the feasible region.
  - A constraint is <u>binding</u> if the LHS and the RHS of the constraint are equal when the optimal values are subbed into the obj. fun. Otherwise, its non<u>binding</u>.

Ex: Reddy Mikks $x_1 = 3$, $x_2 = 1.5$

(1)	$6x_1 + 4x_2 \le 24$	$\longrightarrow$	$18 + 6 = 24$	(Binding)
(2)	$x_1 + 2x_2 \le 6$	$\longrightarrow$	$3 + 3 = 6$	(Binding)
(3)	$-x_1 + x_2 \le 1$	$\longrightarrow$	$-3 + 1.5 = -1.5 < 1$	(Not Binding)
(4)	$x_2 \le 2$	$\longrightarrow$	$1.5 < 2$	(Not Binding)

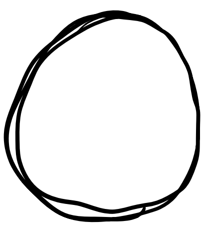
- A set of points S is <u>convex</u> if the line segment connecting any two points in S is also in S. (boundary included)

Ex:

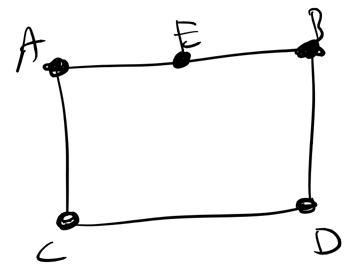


- For any convex set S, a point P in S is an <u>extreme point</u> if each line segment that lies completely in S and contains the point P has P as an endpoint of the line segment.

Ex:



Every point on circumference is an extreme point.



A,B,C,D are extreme
E is not.

Note: also called <u>corner</u> points

- A few facts about LP-problems:

  1.) All iso<u>profit</u> (or iso<u>cost</u>) lines are parallel and a direction of increase (or decrease) can be identified.

  2.) The feasible region of an LP-Problem is <u>convex</u>.

  3.) Any LP that has an optimal solution has an extreme that is optimal.

  - Reduces the set of possible optimal solutions from an infinite set to a finite set.

- With this information, the general technique of the graphical problem is to determine which corner point is optimal
  ↳ motivation for Simplex Method algorithm

WS #3 Solve the minimization Problem.

Section 3.3 : Special Cases

- Four special cases may arise in LP Problems. We'll study them graphically in the Two-Variable Case:

1.) Degeneracy: usually a redundant constraint, an optimal value is 0.

2.) Alternative Optima: infinite # of solutions, isoprofit/cost line is parallel to a constraint.

3.) Unbounded Solutions: direction of increase is in the same direction of unboundedness of feasible region.

4.) No Solution: only infeasible solutions exist, feasible region is empty.

WS #1 – #2 Degenerate and Alternative Optima

WS #1 – #2 Unbounded and Infeasible.

Sections 3.4–3.5: Cost Minimization & Scheduling Problems

- Notes are included on worksheets.

Section 3.6: Investment Problems

- For any investment strategy that may involve a series of cash flows (either negative in the sense of a cash outlay or positive in the sense of a guaranteed return) over a number of years, it is important to calculate the net present value of the investment as a whole.

- Given any investment timeline that details the cash flow over a number of years, as in the table below,

Time	Year 0	Year 1	Year 2	...	Year k	...	Year n
Cash Flow Amount	C_0	C_1	C_2	...	C_k	...	C_n

the value of any cash flow C_k, k years from now, may be different than the present time. Indeed, for many investments (such as a money market fund), any \$1 invested now is actually \$$(1+r)$ one year later (this is guaranteed), where r is the annual interest rate.

- In short, suppose we have \$1000 now, then

Now → $t=0$: $\boxed{1000}$

$t=1$: $1000 + r(1000) \quad = 1000(1+r)$

$\underline{t=2}$: $1000(1+r) + r(1000)(1+r) = 1000(1+r)(1+r) = 1000(1+r)^2$

$\underline{t=3}$: $1000(1+r)^2 + r(1000)(1+r)^2 = 1000(1+r)^2(1+r) = 1000(1+r)^3$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

k years later $\rightarrow$ $\underline{t=k}$: $\boxed{1000(1+r)^k}$

- Thus, in any investment strategy where we are trying to make a decision now (in Year 0) about the evaluation of an investment later, we must incorporate this change in value of a monetary amount due to the interest rate. This is the basis of NPV:

Now		k years later
\$ 1000	=	\$ $1000(1+r)^k$

$$\Rightarrow \quad \frac{\$1000}{(1+r)^k} = \$1000$$

- Thus, if we have a cash flow of \$1000 k years down the road, the present value of that cash flow now is actually less, since it is $\frac{\$1000}{(1+r)^k}$.

- For any investment timeline (as in the table above) we can calculate the present value of each cash flow amount to make an informed decision:

Time	Year 0	Year 1	Year 2	...	Year k	...	Year n
Cash Flow Amount	C_0	C_1	C_2	...	C_k	...	C_n
Present Value	C_0	$\frac{C_1}{(1+r)}$	$\frac{C_2}{(1+r)^2}$	...	$\frac{C_k}{(1+r)^k}$	...	$\frac{C_n}{(1+r)^n}$

- Thus, calculating the Net Present value of the investments amounts to summing up the present values:

$$\boxed{NPV = C_0 + \sum_{k=1}^{n} \frac{C_k}{(1+r)^k}}$$

This allows us to make a judgement call about the actual value of the investment at the current time. All cash flows are in "Time 0 dollars". This is called discounting cash flows to time 0.

- In contrast, we could also compute the net cash flow:

$$[\text{Net Cash Flow}] = C_0 + \sum_{k=1}^{n} C_k$$

but this is a bad assumption as we know the dollar amount changes every year.

$\boxed{\text{WS}}$ #1 working with the NPV formula.

- Note: For an NPV > 0, the investment will add value to the company. Otherwise, it may prove costly.
- With this idea of NPV, we can consider how limited investment funds can be allocated to investment projects.
- In these problems, we may want to only invest in a project partially. That is, we may only purchase a fraction of an investment. Here, we assume the proportionality assumption holds.

Ex: Suppose an investment requires the following cash outlays with the associated NPV:

	Cash Outlays time 0	Cash Outlays time 1	Return NPV
Investment	-1000	-2000	4000

Proportionality assumption: If we make only a $\frac{1}{2}$ commitment in the investment, the cash outlays and NPV follow the same proportion

	Cash Outlays time 0	Cash Outlays time 1	Return NPV
$\frac{1}{2}$ Investment	-500	-1000	2000

- Note: This is often not the case – it may be impossible to purchase a fraction of an investment without sacrificing the investments favorable cash flows. Typically cast this into binary problem (0 or 1, invest or not).

WS #2 & #3 working with investment problems.

Section 3.8: Blending Problems

- Wide array of applications:
  - Crude Oil to make gasoline
  - Mixing various ingredients to produce hot dog meat.
  - Blend various chemicals to make other chemicals.

- Typical Objective Function:
  - Minimize cost
  - Maximize profit
- Constraints
  - Specification of proportions.
  - Demand of type of product
  - Availability / Capacity.

Section 3.10: Multi-Period Production Period.

- In a typical multi-period production problem, we're trying to min costs / max profits over a finite set of time periods (e.g. months, quarters, years etc).
- Goal is to meet the demand for each period while optimizing any leftovers for the next period.
- In a multiperiod model, it is beneficial to define the following variables:

production variables

X_t = # of products produced during time period t

i_t = inventory (goods/products on hand) at the end of year t.

In general, the constraints for i_t can be formulated as follows:

$$i_t = \overbrace{(\text{inventory from previous time period})}^{i_{t-1}} + \dots$$

$$\overbrace{(\text{units produced in period } t)}^{x_t} - \dots$$

$$(\text{units sold in / demand for period } t)$$

- If back-logging is not an option then the sign restriction

$$i_t \geq 0$$

ensures we meet demand on time.

- All other constraints come from the production process.